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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1233

CONTRIBUTION TO THE PROBLEM OF FLOW AT HIGH SPEED

By C. Schmieden and K. H. Kawalki

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OUTLINE

Part I. A Few General Remarks Covering the Prandtl-Busemann Method

Part II. Effect of Compressibility in Axially Symmetrical Flow around
an Ellipsoid

PRELIMINARY REMARKS

The authors regret that due to lack of time the following investigations could not be carried out to a more finished form. Especially in the first part it was intended to include a few further applications and to use them in the general considerations of this part. In spite of the fact that the intentions of the authors could not be realized, the authors felt that it would serve the aims of the competition to present part I in its present fragmentary form.

*Beitrage zum Umströmungsproblem bei hohen Geschwindigkeiten." Lilienthal-Gesellschaft für Luftfahrtforschung Bericht S 13/1. Teil, 1942, pp. 40-68. (Figures referred to in Part I are found immediately after the appendix to that part.)

PART I. A FEW GENERAL REMARKS
COVERING THE PRANDTL-
BUSEMANN METHOD

INTRODUCTION

For the solution of the problem of flow in the subsonic range, two approximation processes are available, the Janzen-Rayleigh method which, proceeding from the potential or the stream function, represents the velocities in form of a progressive series of powers of the Mach number, and the Prandtl-Busemann method which, based upon the fundamental concept of the Prandtl rule, determines the velocities by an expansion in series according to a geometric parameter characterizing the body (references 1 and 2). Both methods supplement each other most opportunely insofar as the first is suited particularly for thick bodies, hence for relatively small critical Mach numbers, while the second method works best for slender body contours whose critical Mach numbers are close to unity. Little is known so far about the limit of convergence of the employed series expansion of both methods; no proof has as yet been given concerning the ceasing of convergence upon the reaching of local sonic velocity; however, it appears plausible from the construction of the, at times very small, number of the explicitly computed terms of the series. The first section of the present report utilizes the long-known relation between velocity and stream density (ρV) in order to prove that with the use of the stream function the differential equations, which serve as basis for the practical calculation in both methods, lose their meaning if the local sonic velocity is exceeded anywhere in the field of the flow. The fact that the solutions of these differential equations obtained by iteration then become useless also, is not surprising. In section 2 it is shown that the expression employed by Hantzche and Wendt for the stream function is not general enough and therefore is likely to fail under certain circumstances, as for instance, for the flow around an ellipse. The latter is discussed in detail and on the basis of the results obtained the suppositions are given and made plausible concerning the fact that the series expansion of the velocities computed by the Janzen-Rayleigh, or the Prandtl-Busemann method can be made to agree by formal transformation. Finally, in the concluding section, the arguments of the first two sections are evaluated for the problem of convergence of the series expansion built upon the potential, with special reference to a report by Görtler (reference 6).

1. If one desires to treat a compressible flow with the differential equation

$$\psi_{xx}(a^2 - u^2) + \psi_{yy}(a^2 - v^2) - 2\psi_{xy}uv = 0 \quad (1)$$

of the corresponding stream function in the two-dimensional case, the quantities a , u , v must be represented as functions of the derivatives of ψ so as to obtain a differential equation containing only ψ and its derivatives. In the analogous case of the potential this transformation is very simple, since owing to

$$u = \phi_x \quad v = \phi_y$$

one immediately obtains

$$\left. \begin{aligned} \phi_{xx}(a^2 - \phi_x^2) + \phi_{yy}(a^2 - \phi_y^2) - 2\phi_x\phi_y\phi_{xy} &= 0 \\ a^2 &= a_0^2 - \frac{\kappa - 1}{2}(\phi_x^2 + \phi_y^2) \end{aligned} \right\} \quad (1a)$$

whereby this equation is obviously valid for all physically possible velocities. But the use of the stream function in this instance involves a characteristic difficulty which becomes most readily apparent through elimination of the sonic velocity from equation (1). If n is the direction of the normal in a point of a stream line, the relation

$$D \equiv \rho V = \frac{\partial \psi}{\partial n} \quad (2)$$

exists between stream function ψ , velocity V , and stream density ρV , while for adiabatic flow

$$\frac{\rho}{\rho_0} = \left[1 - \frac{\kappa - 1}{2} \left(\frac{V}{a_0} \right)^2 \right]^{\frac{1}{\kappa - 1}} \quad (3)$$

is valid.

From (3) it is seen that ρ/ρ_0 decreases monotonically, with increasing velocity; substituting (3) into (2) it is apparent that, while D increases at first with increasing velocity, that on passing through the critical velocity - $V = a^*$ - it reaches a maximum and in the supersonic range for V approaching V_{\max} it decreases to zero again. The approximate course of D against V is shown in figure 1. To effect the required elimination of a , V is used as function of D or $\frac{\partial \psi}{\partial n}$, that is, the inverse of the above function $D(V)$, which will be, in order to be able to make practical calculations, the inverse of the form of a power series

$$V = \underline{P}(D) \quad (4)$$

The problem is thus the following:

Given: a function $D(V)$

Required: the inverse of this function in form of a power series (4), such that $V = 0$ for $D = 0$.

This problem, as known, is solved by the Lagrange inversion formula. This Lagrange series converges in a circular disk with the radius D_{\max} ; in this case D_{\max} equals the value of D , for which

$\frac{dD}{dV} = 0$. This result, which is also clear from inspection of the sketch, naturally holds also for the case where the inverse function is not developed at $D = 0$ but rather at some other point between 0 and D_{\max} , as is done in the Prandtl-Busemann method. The radius of convergence is then correspondingly reduced such that the limit of convergence coincides with sonic boundary. The same holds for the inverse functions u and v as functions of $\frac{\partial \psi}{\partial x}$

and $\frac{\partial \psi}{\partial y}$ which are necessary for the transformation of (1), although in this case the relations are not as clear as for sonic velocity. If one substitutes such power developments in (1) the validity of the thus obtained differential equation for ψ ceases on reaching sonic velocity. Consequently, one should not expect the iterative solutions of the differential equations to have significance once the sonic boundary is exceeded in them. The same applies to the Rayleigh method when it uses the stream function, as well as to the Hantzsche-Wendt method, (developed on the suggestion of Busemann) which improves the Prandtl rule by successive approximation. Both methods use the expansion in powers of the velocity components, hence must fail on reaching the sonic boundary.

It should be interesting to study this failure for an actual single case. Consider the flow around the ellipse at zero incidence for which the Busemann correction is computed in appendix I, equation (20). Even if the stream function computed there did retain its validity when a supersonic zone, no matter how small, has developed about the end point of the small axis,

the curve obtained by plotting $\frac{\partial \psi}{\partial n}$ against the circumference of the ellipse would have to show a course as sketched in figure 2, because of the relation between V and $\frac{\partial \psi}{\partial n}$, sketched in figure 1.

The quantity $\frac{\partial \psi}{\partial n}$ would have to attain its maximum at the beginning of the supersonic zone and then decrease monotonously to the end point of the small axis. Actually $\frac{\partial \psi}{\partial n}$ shows an entirely different course (fig. 3). It increases monotonically with the arc length for a supercritical Mach number and numerically remains always smaller than $\left(\frac{\partial \psi}{\partial n}\right)_{\max}$. But to such a $\frac{\partial \psi}{\partial n}$ distribution there belongs, according to figure 1, for reasons of continuity, either a pure subsonic or a pure supersonic flow, both of which are impossible at a supercritical Mach number and in the presence of stagnation points.

In figure 4 the stream-density distribution $\left(\frac{\partial \psi}{\partial n} \text{ distribution}\right)$ for an ellipse of axis ratio 1/10 at three different Mach numbers

is plotted against the conventional parameter angle θ , that is only the range of stream-density values of interest here. It is seen that the curve for the subcritical Mach number 0.80 runs below the curve for 0.85, in the vicinity of the end point of the small axis, hence, that the stream density increases as yet with rising Mach number. $M = 0.85$ is almost exactly the critical Mach number, consequently, the solution must fail at still larger Mach numbers. As a matter of fact, there results for $M = 0.90$ the above sketched behavior of the $\frac{\partial \psi}{\partial n}$ distribution; the solution is impractical.

Figure 5 serves to illustrate the good convergence of the Busemann method in the range of its validity. For the same ellipse as in figure 4 the stream density at the end point of the minor axis is again plotted as function of the Mach number in the range of interest; the upper curve holds for the first step of the method, the Prandtl rule, the lower curve for the second approximation. The intersection of these curves with the straight line of the sonic boundary gives the critical Mach number in the corresponding approximation (hence $M_{cr} = 0.74$ and 0.78).

In connection with the foregoing arguments, however, another value is of greater interest, which can also be taken from these curves. The maximum stream density in first approximation has the

value $0.598 \frac{\rho V_t}{\rho_0 a_0}$, in the second approximation the value $0.583 \frac{\rho V_t}{\rho_0 a_0}$,

whereas the true value naturally coincides with the sonic boundary,

hence must be $0.578 \frac{\rho V_t}{\rho_0 a_0}$. Thus, since the rigorous solution should

yield a stream-density distribution which would have to approach the sonic boundary from below, at a critical Mach number of around 0.61, the plot indicates that the difference between the first approximation and the rigorous solution is largely canceled out by the second step of the method, so that the method converges quite well.

Moreover, it is pointed out that, at variance with the customary terminology in the discussion of figure 4, the Mach number used as critical Mach number is that corresponding to the maximum of the stream density; for up to that Mach number the solution obtained by this approximation has a meaning.

2. The Hantzsche-Wendt calculations proceed from the following assumption. If λ is a parameter that characterizes the departure of the profile form from a straight section profile (for example, thickness, camber, angle of attack) at zero incidence the stream function can be represented by a power series in powers of λ , the coefficients of which are functions of x and y :

$$\frac{\psi}{U} = y + g(x_1 y) \lambda + h(xy) \lambda^2 + \dots \quad (1)$$

On the basis of this assumption it is possible to set up, by means of comparison of coefficients, a recursive system of differential equations for the functions of the individual powers of λ which then can be integrated as in the Janzen-Rayleigh method.

But while in the Rayleigh method the expansion of the stream function in powers of M^2

$$\psi = \psi_0 + M^2 \psi_1 + M^4 \psi_2 + \dots \quad (2)$$

is always possible, the expansion of ψ in the form (1) is by no means guaranteed a priori. On the contrary it will be shown that it is not possible at least in the two cases cited here, the ellipse and the ellipsoid of revolution both at zero incidence, but that sooner or later terms of the form $\lambda^v \ln \lambda$ must appear in these cases. If that is the case it ceases to be possible to set up the recursive system of equations as was done by Hantzsche-Wendt. The best procedure is then as follows: The stream function is put in the form

$$\frac{\psi}{U} = \psi_0(x_1 y) + \psi_1(x_1 y) + \psi_2(x_1 y) + \dots \quad (3)$$

ψ_0 is the stream function in the undisturbed free stream, $\psi_0 + \psi_1$ represents the Prandtl approximation; the functions ψ_v with $v \geq 2$ which are assumed to satisfy the boundary condition $\psi_v = 0$ on the contour, are regular in the outside domain and at infinity

their derivatives vanish to a sufficient degree. It is further assumed that, apart from sufficiently small zones, especially the immediate vicinity of stagnation points, the function ψ_0 and its derivatives are small compared to all ψ_K and their derivatives with subscripts $K < 0$. The question whether this assumption is satisfied can be answered only after calculation of ψ_0 .

The fact that the concept of the order of magnitude is a little vague, especially since the vicinity of the stagnation points must be excluded, lies in the nature of the problem and seems therefore unavoidable; besides no difficulties arose in the calculation of actual cases. When limited to the second approximation the new rule gives the same result in all the cases treated by Hantzsche-Wendt, in particular, the expansion parameters assumed in (1) are automatically obtained. But if ψ can no longer be represented by a power development of the form (1), the differently constructed terms are covered by our formula. The necessity of excluding the stagnation point vicinity from the appraisal of the order of magnitude lies in the initial step of the method, the Prandtl rule, and has been voiced often enough as principal objection against this rule when stagnation points exist. More accurate numerical checks (reference 3) have shown, however, that the Prandtl rule gives a fairly close approximation even in the vicinity of the stagnation points, although its assumptions are by no means satisfied any longer. It is not believed that this behavior is due to a lucky accident. However, the following supposition (unfortunately without proof) seems reasonable:

If it were possible to compute the series (2) to any high 0 , this series would in all probability converge absolutely, but not uniformly and would merely represent a rearrangement of the Rayleigh series computed for the same contour to any high powers of Mach number.

In other words, if each term in (2) is developed in powers of the Mach number the formal rearrangement in series of these powers gives the Rayleigh series.

If this supposition is correct, the Prandtl rule would be legitimate also for the vicinity of the stagnation point in a certain manner and would explain the surprisingly good approximation of the rule at these points. Unfortunately there is very little material available to test this conjecture, which for lack of time could be evaluated only in two cases, the ellipse and the circular-arc profile with shock-free entry.

In both cases the result of the examination was positive; however, only the more interesting case of the ellipse is treated here.

The Rayleigh approximation up to the term with M^2 inclusive is known (reference 4). The end point of the small axis, that is, the maximum speed is

$$\frac{V_{\max}}{U} = \frac{1}{1 + \sigma^2} \left\{ 2 + \frac{M^2}{2} \left[\frac{1 - \sigma^2}{\sigma^2} - \epsilon^4 \frac{(1 + \sigma^2)^2}{2\sigma^4} \ln \frac{1 + \sigma^2}{1 - \sigma^2} + \epsilon^3 \frac{1 + \sigma^2}{\sigma^3} \arctan \frac{2\sigma}{1 - \sigma^2} - \frac{2\epsilon^2}{\sigma^2} \right] \right\}, \quad \epsilon = \frac{1 - \sigma^2}{1 + \sigma^2} \quad (4)$$

The development in powers of ϵ gives, up to the terms with ϵ^4

$$\frac{V}{U} = 1 + \epsilon + \frac{M^2}{2} \left[\epsilon + \epsilon^2 + \epsilon^3 \left(\frac{\pi}{2} - 1 \right) + \epsilon^4 \ln \epsilon \right] \quad (5)$$

On the other hand the second approximation computed by (1) or (3) gives in both cases, if one develops in terms of Mach number and breaks off with the term in M^2 ,

$$\frac{V_{\max}}{U} = 1 + \epsilon + \frac{M^2}{2} (\epsilon + \epsilon^2) \quad (6)$$

so that the terms, which are for ϵ as well as for M are at the most of the second degree, will be contained in both formulas. The differences in the definition of the Mach number - Rayleigh refers to the velocity of sound at rest formulas (1) and (3) to the free-stream sonic velocity - do not make themselves felt as yet in this approximation.

The third approximation in the appendix of the Busemann method based on (3) was carried only far enough to show the appearance of a term $\epsilon^4 \ln \epsilon$. It was not further carried out because of the prohibitive amount of calculations required in order to obtain the final formulas from the final formula given there; it can be seen that a term $\epsilon^3 \frac{\pi}{2}$ must appear too, which

likewise implies that agreement of both developments is to be expected for the terms $\sim M^2 \epsilon^4 \ln \epsilon$ and $\sim M^2 \epsilon^3$.

3. The following conclusions are drawn with the use of the stream function. The Prandtl-Busemann and the Janzen-Rayleigh method give identical results, but only when both developments are carried out completely. Each development represents a rearrangement of the other. Both developments diverge if at any point in the flow domain the local velocity of sound is exceeded.

But in both cases the velocity potential could be used instead of the stream function, as is, in fact, done predominantly in the Rayleigh method. However, no case of flow past a body has been calculated as yet with the Prandtl-Busemann method. One of the likely reasons for this might be found in the complicated nature of the boundary condition for the potential. Since the reduction of the differential equation to coefficients dependent on Φ alone is in this instance easily achieved for the entire physically possible speed range (as mentioned already in (1a)), no absolutely valid reason for a ceasing of convergence of the solution on passing through the velocity of sound can be found from the differential equation alone, owing to its nonlinearity. But, since the values of the velocity computed for identical conditions from the stream function and the potential are identical in the domain of convergence of the stream function, there is a strong suspicion that the potential development itself ceases to converge at the same place. In fact, all the calculation made by the Rayleigh method with the potential, particularly those by Lamla (reference 5) for circle and sphere, indicate that the obtained series cease to converge on reaching the sonic boundary, so far as such a conclusion can be drawn at all from the few explicitly known terms of these series. Consolidating this result with those obtained above for the stream function it may be stated with great probability that even the fourth method, namely, the Prandtl-Busemann method which uses the potential, has the same limit of convergence as the other three. This statement contradicts the result of a report by Görtler (reference 6); therefore, a brief critical review of his report will be given consequently.

Görtler used his method to compute the flow past a wavy wall where supersonic zone of finite extent occurs. Against this result of Görtler some serious objections may be made.

1. The problem of convergence of his development, of which the first three terms are explicitly calculated, is practically ignored.

2. A part of the coefficients of his first example is wrong, according to a careful check and with the correct coefficients the convergence becomes definitely worse.

3. The curves of figures 1(a) and 1(b) in his report are constructed with the wrong coefficients. Whether the correct values of the constants already bring the supersonic zone, small by itself, to disappearance could not be checked, unfortunately.

This is not the place to be polemical concerning Görtler's report. That was not our intention. However, the doubts have to be brought to attention, for his report does not give a proof, that with the development followed therein a solution could be obtained, whose range of validity would extend to a small region of the supersonic domain.

APPENDIX

CALCULATION OF THE THIRD APPROXIMATION FOR THE ELLIPSE

In the general differential equation for the stream function

$$\psi_{xx}(a^2 - u^2) + \psi_{yy}(a^2 - v^2) - 2uv\psi_{xy} = 0 \quad (1)$$

it is necessary, in accordance with the Prandtl-Busemann method of the splitting off of the free-stream part U parallel to the x -axis, to set

$$u = U + u_1; \quad v = v_1 \quad (2)$$

where the quantities carrying the subscript 1 denote disturbance quantities. The following is then rigorously valid (a_1 = velocity of sound of free stream U)

$$\left. \begin{aligned} a^2 &= a_1^2 - (\kappa - 1) U u_1 - \frac{\kappa - 1}{2} (u_1^2 + v_1^2) \\ u^2 &= U^2 + 2U u_1 + u_1^2; \quad v^2 = v_1^2 \end{aligned} \right\} \quad (3)$$

We write for abbreviation

$$M = \frac{U}{a_1}; \quad \mu = \frac{1}{\sqrt{1 - M^2}} \rightarrow \mu^2 M^2 = \mu^2 - 1 \quad (4)$$

Equation (3) substituted in (1) gives

$$\begin{aligned} &\psi_{xx} \left[a_1^2 - U^2 - (\kappa + 1) U u_1 - \frac{\kappa + 1}{2} u_1^2 - \frac{\kappa - 1}{2} v_1^2 \right] \\ &+ \psi_{yy} \left[a_1^2 - (\kappa - 1) U u_1 - \frac{\kappa - 1}{2} u_1^2 - \frac{\kappa + 1}{2} v_1^2 \right] \\ &- 2\psi_{xy}(U v_1 + u_1 v_1) = 0 \end{aligned} \quad (5)$$

The abbreviation $\lambda = \frac{\rho_1}{\rho_0}$; $v = \frac{1}{\lambda}$ gives for the subsonic zone in second approximation the relation between the disturbance velocities and the disturbance stream function ψ_1 . (Owing to $\psi_{xx} = \psi_{lxx}$ etc. the subscript 1 for ψ_1 is omitted hereafter as there exists no danger of confusion.)

$$u_1 = v \mu^2 \psi_y + \frac{U v^2 \mu^2}{2 a_1^2} \left\{ \psi_y^2 \left[\mu^4 (\kappa + 1) + \mu^2 (2 - \kappa) \right] + \psi_x^2 \right\}$$

$$u_1^2 = v^2 \mu^4 \psi_y^2 \quad (6)$$

$$v_1 = -v \psi_x - \frac{U v^2 \mu^2}{a_1^2} \psi_x \psi_y$$

$$v_1^2 = v^2 \psi_x^2$$

Dividing (5) by a_1^2 and moving all terms of higher order to the right gives

$$\psi_{xx}(1 - M^2) + \psi_{yy} = \psi_{xx} \left[(\kappa + 1) \frac{U u_1}{a_1^2} + \frac{\kappa + 1}{2} \frac{u_1^2}{a_1^2} + \frac{\kappa - 1}{2} \frac{v_1^2}{a_1^2} \right]$$

$$+ \psi_{yy} \left[(\kappa - 1) \frac{U u_1}{a_1^2} + \frac{\kappa - 1}{2} \frac{u_1^2}{a_1^2} + \frac{\kappa + 1}{2} \frac{v_1^2}{a_1^2} \right] + 2 \psi_{xy} \left(\frac{U v_1}{a_1^2} + \frac{u_1 v_1}{a_1^2} \right) \quad (7)$$

(6) substituted in (7) gives

$$\psi_{xx}(1 - M^2) + \psi_{yy} = \frac{v U}{a_1^2} \left[(\kappa + 1) \mu^2 \psi_y \psi_{xx} + (\kappa - 1) \mu^2 \psi_y \psi_{yy} \right.$$

$$\left. - 2 \psi_x \psi_{xy} \right] + \psi_{xx} \frac{v^2}{2 a_1^2} \left\{ (\kappa + 1) \psi_y^2 \left[\mu^6 (\kappa + 1) - 2 \mu^4 (\kappa - 1) \right. \right.$$

$$\left. \left. - \mu^2 (2 - \kappa) \right] + \psi_x^2 \left[(\kappa + 1) (\mu^2 - 1) + \kappa - 1 \right] \right\}$$

$$+ \psi_{yy} \frac{v^2}{2 a_1^2} \left\{ (\kappa - 1) \psi_y^2 \left[\mu^6 (\kappa + 1) - 2 \mu^4 (\kappa - 1) - \mu^2 (2 - \kappa) \right] \right.$$

$$\left. + \psi_x^2 \left[(\kappa - 1) (\mu^2 - 1) + \kappa + 1 \right] \right\} - 2 \psi_{xy} \psi_x \psi_y \frac{v^2}{a_1^2} (2 \mu^2 - 1) \quad (8)$$

The flow pattern is now affinely distorted in the direction normal to the flow by the Prandtl transformation

$$x = \xi; y = \mu\eta; \rightarrow \psi_x = \psi_\xi; \psi_y = \frac{1}{\mu} \psi_\eta; \psi_{yy} = \frac{1}{\mu^2} \psi_{\eta\eta}; \psi_{xy} = \frac{1}{\mu} \psi_{\xi\eta}$$

we further put

$$\psi = \lambda U \mu \psi^*$$

and immediately omit the asterisk for ψ , after which (8) gives by multiplication with μ^2

$$\begin{aligned} \Delta\psi = \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = (\mu^2 - 1) & \left[(\kappa + 1) \mu^2 \psi_\eta \psi_{\xi\xi} + (\kappa - 1) \psi_\eta \psi_{\eta\eta} \right. \\ & \left. - 2 \psi_\xi \psi_{\xi\eta} \right] + \frac{1}{2} (\mu^2 - 1) \left(\mu^2 \psi_{\xi\xi}^2 \left[(\kappa + 1) \psi_\eta^2 \left[\mu^4 (\kappa + 1) \right. \right. \right. \right. \\ & \left. \left. \left. - 2\mu^2 (\kappa - 1) - (2 - \kappa) \right] + \psi_\xi^2 \left[\mu^2 (\kappa + 1) - 2 \right] \right] \right. \\ & \left. + \psi_{\eta\eta} \left\{ (\kappa - 1) \psi_\eta^2 \left[\mu^4 (\kappa + 1) - 2\mu^2 (\kappa - 1) - (2 - \kappa) \right] \right. \right. \\ & \left. \left. + \psi_\xi^2 \left[\mu^2 (\kappa - 1) + 2 \right] \right\} - 4 \psi_{\xi\eta} \psi_\xi \psi_\eta (2\mu^2 - 1) \right) \end{aligned}$$

Next we put $\psi = \psi_1 + \psi_2 + \psi_3 + \dots$ and determine $\psi_1, \psi_2, \psi_3 \dots$ successively from (9) on the assumption that ψ is small compared to ψ_0 (free stream) ψ_2 small relative to ψ_1, ψ_3 small relative to ψ_2 and so on. If the flow involves a body with stagnation points, this assumption is certainly not fulfilled in the vicinity of the stagnation points; in such a case it must suffice to postulate the validity of the assumption outside of a certain range from the stagnation point and to check the justification for the assumption on hand of the result of the calculation. Discounting this difficulty for the present, the

conventional iteration method can be used in our case as it was used in the Rayleigh method. Thus it obtains for ψ_1 and ψ_2 the following two differential equations

$$\Delta \psi_1 = 0; \quad \Delta \psi_2 = (\mu^2 - 1)$$

$$\left[(\kappa + 1) \mu^2 \psi_{1\eta} \psi_{1\xi\xi} + (\kappa - 1) \psi_{1\eta} \psi_{1\eta\eta} - 2 \psi_{1\xi} \psi_{1\xi\eta} \right] \quad (9a)$$

Substituting ψ_1 in the remaining terms of the right-hand side of (9) gives a part of the right-hand side for $\Delta \psi_3$; the rest of the disturbance terms belonging to $\Delta \psi_3$ are computed later. Because $\psi_{1\eta\eta} = -\psi_{1\xi\xi}$ the first part of the disturbance terms becomes

$$\frac{1}{2} (\mu^2 - 1) \left(\psi_{1\xi\xi} \left\{ \psi_{1\eta}^2 \left[(\kappa + 1) \mu^2 - (\kappa - 1) \right] \left[\mu^4 (\kappa + 1) - 2 \mu^2 (\kappa - 1) - (2 - \kappa) \right] + \psi_{1\xi}^2 \left[\mu^2 (\mu^2 - 1) (\kappa + 1) - 2 \right] \right\} \right. \\ \left. - 4 \psi_{1\xi\eta} \psi_{1\xi} \psi_{1\eta} (2 \mu^2 - 1) \right) \quad (10)$$

Because $\Delta \psi_1 = 0$, ψ_1 can be regarded as imaginary part of an analytical function $w(\zeta)$ $\zeta = (\xi + i\eta)$:

$$w = \phi_1 + i \psi_1; \quad w' = \frac{dw}{d\zeta} = \psi_{1\eta} + i \psi_{1\xi}$$

$$w'' = \frac{d^2 w}{d\zeta^2} = \psi_{1\xi\eta} + i \psi_{1\xi\xi}$$

On denoting conjugate complex quantities by a bar, all the combinations of derivatives of ψ_1 appearing above can be represented

by imaginary parts of functions of ξ and $\bar{\xi}$: $w' = \frac{dw}{d\xi}$

$$\left. \begin{aligned} 2 \psi_{1\xi\xi} \psi_{1\eta} &= J (w'' w' + w'' \bar{w}') \\ 2 \psi_{1\xi\eta} \psi_{1\xi} &= J (w'' w' - w'' \bar{w}') \\ 4 \psi_{1\xi\xi} \psi_{1\eta}^2 &= J \left[w'' (w' + \bar{w}')^2 \right] \\ 4 \psi_{1\xi\xi} \psi_{1\xi}^2 &= -J \left[w'' (w' - \bar{w}')^2 \right] \\ 4 \psi_{1\xi\eta} \psi_{1\xi} \psi_{1\eta} &= J \left[w'' (w'^2 - \bar{w}'^2) \right] \end{aligned} \right\} \quad (11)$$

Substituting (11) in (10) gives

$$\Delta_1 \psi_3 = J \left(A w'' w'^2 + 2 B w'' w' \bar{w}' + C w'' \bar{w}'^2 \right) \quad (12)$$

where

$$A = \frac{1}{8} (\mu^2 - 1) (\alpha - \beta - \gamma), \quad B = \frac{1}{8} (\mu^2 - 1) (\alpha + \beta)$$

$$C = \frac{1}{8} (\mu^2 - 1) (\alpha - \beta + \gamma)$$

and

$$\alpha = \left[(\kappa + 1) \mu^2 - (\kappa - 1) \right] \left[\mu^4 (\kappa + 1) - 2 \mu^2 (\kappa - 1) - (2 - \kappa) \right],$$

$$\beta = \mu^2, (\mu^2 - 1), (\kappa + 1) - 2 \quad \gamma = 4 (2 \mu^2 - 1)$$

With the abbreviation

$$\sigma = (\kappa + 1) (\mu^2 - 1)$$

the expressions for α , β , γ and A , B , C can be further consolidated

$$\alpha = (\sigma + 2) \left[(\mu^2 - 1) \sigma + 4 \mu^2 - 3 \right]$$

$$\beta = \mu^2 \sigma - 2; \gamma = 4 (\mu^2 - 1)$$

hence

$$\left. \begin{aligned} A &= \frac{1}{8} (\mu^2 - 1)^2 \sigma (\sigma + 5) \\ B &= \frac{1}{8} (\mu^2 - 1) \left[(\mu^2 - 1) \sigma^2 + (7 \mu^2 - 5) \sigma + 8 (\mu^2 - 1) \right] \\ C &= \frac{1}{8} (\mu^2 - 1) \left[(\mu^2 - 1) \sigma^2 + 5 (\mu^2 - 1) \sigma + 8 (\mu^2 - 1) \right] \end{aligned} \right\} (13)$$

These terms are now supplemented by those obtained from the products of ψ_1 and ψ_2 after insertion in the first bracket on the right-hand side of (9); they are

$$\begin{aligned} & (\mu^2 - 1) \left[(\kappa + 1) \mu^2 (\psi_{1\eta} \psi_{2\xi\xi} + \psi_{2\eta} \psi_{1\xi\xi}) \right. \\ & \quad + (\kappa - 1) (\psi_{1\eta} \psi_{2\eta\eta} + \psi_{2\eta} \psi_{1\eta\eta}) \\ & \quad \left. - 2 (\psi_{1\xi} \psi_{2\xi\eta} + \psi_{2\xi} \psi_{1\xi\eta}) \right] \\ & = (\mu^2 - 1) \left\{ \psi_{2\eta} \psi_{1\xi\xi} \left[(\kappa + 1) \mu^2 - (\kappa - 1) \right] \right. \\ & \quad - 2 \psi_{2\xi} \psi_{1\xi\eta} + \psi_{1\eta} \left[(\kappa + 1) \mu^2 \psi_{2\xi\xi} + (\kappa - 1) \psi_{2\eta\eta} \right] \\ & \quad \left. - 2 \psi_{1\xi} \psi_{2\xi\eta} \right\} \end{aligned} \quad (14)$$

Now

$$\psi_2 = J (W_{\xi}, \bar{\xi}) \quad (\text{see also (20)})$$

and

$$W_{\xi} = W_{\xi} + W_{\bar{\xi}}; W_{\eta} = i (W_{\xi} - W_{\bar{\xi}}); W_{\xi\eta} = i (W_{\xi\xi} - W_{\bar{\xi}\bar{\xi}})$$

$$W_{\xi\xi} = W_{\xi\xi} + 2 W_{\xi\bar{\xi}} + W_{\bar{\xi}\bar{\xi}}; W_{\eta\eta} = - W_{\xi\xi} + 2 W_{\xi\bar{\xi}} - W_{\bar{\xi}\bar{\xi}}$$

Hence

$$\left. \begin{aligned} \psi_{2\xi} &= J (W_{\xi} + W_{\bar{\xi}}); \psi_{2\eta} = J (i W_{\xi} - i W_{\bar{\xi}}) \\ \psi_{2\xi\eta} &= J (i W_{\xi\xi} - i W_{\bar{\xi}\bar{\xi}}); \psi_{2\xi\xi} = J (W_{\xi\xi} + 2 W_{\xi\bar{\xi}} + W_{\bar{\xi}\bar{\xi}}) \\ \psi_{2\eta\eta} &= J (- W_{\xi\xi} + 2 W_{\xi\bar{\xi}} - W_{\bar{\xi}\bar{\xi}}) \end{aligned} \right\} \quad (15)$$

and

$$\left. \begin{aligned} 2\psi_{1\eta} &= w' + \bar{w}'; 2i\psi_{1\xi} = w' - \bar{w}' \\ 2\psi_{1\xi\eta} &= w'' + \bar{w}''; 2i\psi_{1\xi\xi} = w'' - \bar{w}'' \end{aligned} \right\} \quad (15a)$$

Substituting (15) and (15a) in (14) gives

$$\begin{aligned} \Delta_2 \psi_3 &= \frac{\mu^2 - 1}{2} J \left\{ \left[(\kappa + 1) \mu^2 - (\kappa - 1) \right] (W_{\xi} - W_{\bar{\xi}}) (w'' - \bar{w}'') \right. \\ &\quad - 2 (W_{\xi} + W_{\bar{\xi}}) (w'' + \bar{w}'') + \left[(\kappa + 1) \mu^2 - (\kappa - 1) \right] (W_{\xi\xi} + W_{\bar{\xi}\bar{\xi}}) \\ &\quad \left. (w' + \bar{w}') - 2 (W_{\xi\xi} - W_{\bar{\xi}\bar{\xi}}) (w' - \bar{w}') \right. \\ &\quad \left. + 2 \left[(\kappa + 1) \mu^2 + (\kappa - 1) \right] W_{\xi\bar{\xi}} (w' + \bar{w}') \right\} \end{aligned} \quad (16)$$

where it should be noted that (see also further below)

$$4 W_{\xi \bar{\xi}} = \frac{\mu^2 - 1}{2} \left[\sigma w'' w' + (\sigma + 4) w'' \bar{w}' \right] \quad (17)$$

The right-hand side of (16) can be further simplified by the previously used abbreviation σ

$$(\kappa + 1) (\mu^2 - 1) = \sigma; (\kappa + 1) \mu^2 - (\kappa - 1) = \sigma + 2$$

$$(\kappa + 1) \mu^2 + \kappa - 1 = \sigma + 2\kappa$$

and with it the following representation is obtained for $\Delta_2 \psi_3$

$$\begin{aligned} \Delta_2 \psi_3 = \frac{\mu^2 - 1}{2} J \left\{ \sigma (W_{\xi} w'' + W_{\bar{\xi}} \bar{w}'' + W_{\xi \xi} w' + W_{\xi \bar{\xi}} w') \right. \\ \left. - (\sigma + 4) (W_{\xi} \bar{w}'' + W_{\bar{\xi}} w'' - W_{\xi \xi} \bar{w}' - W_{\bar{\xi} \bar{\xi}} w') \right. \\ \left. + \frac{\mu^2 - 1}{4} (\sigma + 2\kappa) (w' + \bar{w}') \left[\sigma w' w'' + (\sigma + 4) w'' \bar{w}' \right] \right\} \quad (18) \end{aligned}$$

Equations (12) and (18) together give the disturbance equation for the third approximation ψ_3 in the sense of our iteration method.

If z is the coordinate of the image plane which contains a circle, ξ the coordinate of the plane of flow, we get with R as radius of the circle

$$\left. \begin{aligned} \xi &= z + \frac{a^2}{z}; \quad w = \frac{R^2 - a^2}{z}; \quad \frac{a^2}{R^2} < 1; \quad \epsilon = \frac{R^2 - a^2}{R^2 + a^2} \\ \frac{d\xi}{dz} &= \frac{z^2 - a^2}{z^2}; \quad \frac{dw}{dz} = -\frac{R^2 - a^2}{z^2}; \quad \frac{dw}{d\xi} = w' = -\frac{R^2 - a^2}{z^2 - a^2} \end{aligned} \right\} \quad (19)$$

The transformation to image-plane circle coordinates gives for ψ_2 and W

$$\Delta \psi_2 = J \left(4 \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} \right) = J \left(4 \frac{\partial^2 W}{\partial z \partial \bar{z}} \frac{dz}{d\xi} \frac{d\bar{z}}{d\bar{\xi}} \right)$$

and

$$4 \frac{\partial^2 W}{\partial z \partial \bar{z}} = \frac{\mu^2 - 1}{2} \left[\frac{\sigma}{2} \frac{d}{dz} \left(\frac{dw}{d\xi} \right)^2 \frac{d\bar{\xi}}{dz} + (\sigma + 4) \frac{d}{dz} \left(\frac{dw}{d\xi} \right) \frac{d\bar{w}}{dz} \right] \quad (19a)$$

hence with the observance of the boundary condition

$$W = \frac{\mu^2 - 1}{8} (R^2 - a^2)^2 \left[\frac{\sigma}{2} \frac{1}{(z^2 - a^2)^2} \left(\bar{z} - \frac{a^2 z}{R^2} + \frac{a^2}{z} - \frac{R^2}{z} \right) - \frac{\sigma + 4}{(z^2 - a^2)} \left(\frac{1}{z} - \frac{z}{R^2} \right) \right] = \frac{\mu^2 - 1}{8} W_1(z, \bar{z}) \quad (20)$$

To reduce the right-hand side of (12) and (18) to image-plane circle coordinates it is borne in mind that

$$W_\xi = W_z \frac{dz}{d\xi}; \quad W_{\xi \xi} = \frac{d}{dz} \left(W_z \frac{dz}{d\xi} \right) \frac{dz}{d\xi}; \quad W_{\bar{\xi}} = W_{\bar{z}} \frac{d\bar{z}}{d\bar{\xi}}; \quad W_{\bar{\xi} \bar{\xi}} = \frac{d}{d\bar{z}} \left(W_{\bar{z}} \frac{d\bar{z}}{d\bar{\xi}} \right) \frac{d\bar{z}}{d\bar{\xi}}$$

Hence, after introducing W_1 from (12) and (13)

$$\begin{aligned}
 4 \frac{\partial^2 W_1}{\partial z \partial \bar{z}} = & A \frac{d}{dz} \left(\frac{w'^3}{3} \right) \frac{d\bar{f}}{dz} + B \frac{d}{dz} (w'^2) \frac{d\bar{w}}{dz} + C \frac{d}{dz} (w') \left(\frac{d\bar{w}}{dz} \right)^2 \frac{d\bar{f}}{dz} \\
 & + \left(\frac{u^2 - 1}{4} \right)^2 \left(\sigma \left[\frac{d}{dz} \left(w' W_{1z} \frac{dz}{d\zeta} \right) \frac{d\bar{f}}{dz} + \frac{d}{d\bar{z}} \left(\bar{w}' W_{1\bar{z}} \frac{d\bar{z}}{d\bar{\zeta}} \right) \frac{d\bar{f}}{d\bar{z}} \right] \right. \\
 & - (\sigma + 4) \left[\frac{d}{d\bar{z}} (\bar{w}) W_{1z} + \frac{d}{dz} (w) W_{1\bar{z}} - \frac{d}{dz} W_{1z} \frac{dz}{d\zeta} \frac{d\bar{w}}{d\bar{\zeta}} \right. \\
 & \left. \left. - \frac{d}{d\bar{z}} (W_{1\bar{z}} \frac{d\bar{z}}{d\bar{\zeta}}) \frac{dw}{dz} \right] + 2 (\sigma + 2\kappa) \left\{ \sigma \left[\frac{d}{dz} \left(\frac{w'^3}{3} \right) \frac{d\bar{f}}{dz} \right. \right. \right. \\
 & \left. \left. + \frac{d}{dz} \left(\frac{w'^2}{2} \right) \frac{d\bar{w}}{dz} \right] + (\sigma + 4) \left[\frac{d}{dz} \left(\frac{w'^2}{2} \right) \frac{d\bar{w}}{dz} + \frac{d}{dz} (w') \frac{d\bar{w}}{dz} \frac{d\bar{f}}{dz} \right] \right\} \right) \quad (21)
 \end{aligned}$$

It is essential to note that expressions such as $w' W_{1z} \frac{dz}{d\zeta}$ are functions of z and \bar{z} , hence must be split up before (21) can be integrated in the usual manner.

The function W_1 can be written (see also (20))

$$W_1 = \Phi_1(z) \bar{\zeta} + \Phi_2(z) \bar{w} + \Phi_3(z)$$

with

$$\begin{aligned}
 \Phi_1(z) &= \frac{\sigma}{2} w'^2; \quad \Phi_2(z) = (\sigma + 4) w' \\
 \Phi_3(z) &= -\frac{\sigma}{2} w'^2 \left(\frac{z^2}{R^2} + \frac{R^2}{z} \right) - (\sigma + 4) w' \frac{R^2 - z^2}{R^2}
 \end{aligned}$$

Hence

$$W_{1z} = \Phi_1'(z) \bar{\zeta} + \Phi_2'(z) \bar{w} + \Phi_3'(z); \quad W_{1\bar{z}} = \Phi_1 \frac{d\bar{\zeta}}{d\bar{z}} + \Phi_2 \frac{d\bar{w}}{d\bar{z}}$$

and

$$W_{1\bar{z}} \frac{\bar{dz}}{d\xi} = \Phi_1 + \Phi_2 \bar{w}'; \quad \frac{d}{d\bar{z}} \left(W_{1\bar{z}} \frac{\bar{dz}}{d\xi} \right) = \Phi_2 \frac{d\bar{w}'}{d\bar{z}}$$

$$\bar{w}' W_{1\bar{z}} \frac{\bar{dz}}{d\xi} = \Phi_1 \bar{w}' + \Phi_2 \bar{w}'^2$$

$$\frac{d}{d\bar{z}} \left(W_{1\bar{z}} \frac{\bar{dz}}{d\xi} \bar{w}' \right) = \Phi_1 \frac{d\bar{w}'}{d\bar{z}} + \Phi_2 \frac{d}{d\bar{z}} (\bar{w}'^2)$$

Correspondingly the explicit expressions for the derivatives used in equation (21)

$$\frac{d}{d\bar{z}} \left(\bar{w}' W_{1\bar{z}} \frac{\bar{dz}}{d\xi} \right) \frac{d\bar{f}}{dz} = \frac{d}{d\bar{z}} (\bar{w}') \frac{\sigma}{2} \frac{d\bar{w}}{dz} \frac{d\bar{w}}{d\xi} + \frac{d}{d\bar{z}} (\bar{w}'^2) (\sigma + 4) \frac{d\bar{w}}{dz}$$

$$\frac{d}{dz} \left(\bar{w}' W_{1\bar{z}} \frac{\bar{dz}}{d\xi} \right) \frac{d\bar{f}}{dz} = \frac{d}{dz} \left(\bar{w}' \Phi_1' \frac{dz}{d\xi} \right) \bar{f} \frac{d\bar{f}}{dz}$$

$$+ \frac{d}{dz} \left(\bar{w}' \Phi_2' \frac{dz}{d\xi} \right) \bar{w} \frac{d\bar{f}}{dz} + \frac{d}{dz} \left(\bar{w}' \Phi_3' \frac{dz}{d\xi} \right) \frac{d\bar{f}}{dz}$$

$$W_{1\bar{z}} \frac{d\bar{w}'}{d\bar{z}} = \Phi_1' \bar{f} \frac{d}{d\bar{z}} (\bar{w}') + \Phi_2' \bar{w} \frac{d\bar{w}'}{d\bar{z}} + \Phi_3' \frac{d\bar{w}'}{d\bar{z}}$$

$$\frac{d}{d\bar{z}} \left(W_{1\bar{z}} \frac{\bar{dz}}{d\xi} \right) \frac{d\bar{w}}{dz} = (\sigma + 4) \frac{d\bar{w}'}{d\bar{z}} \frac{d\bar{w}}{dz} \frac{d\bar{w}}{d\xi}$$

$$W_{1\bar{z}} \frac{d\bar{w}'}{dz} = \Phi_1 \frac{d\bar{w}'}{dz} \frac{d\bar{f}}{dz} + \Phi_2 \frac{d\bar{w}'}{dz} \frac{d\bar{w}}{dz} = \frac{\sigma}{6} \frac{d\bar{w}'}{dz} \frac{d\bar{f}}{dz}$$

$$+ \frac{\sigma + 4}{2} \frac{d\bar{w}'}{dz} \frac{d\bar{w}}{dz}; \quad \frac{d}{dz} \left(W_{1\bar{z}} \frac{\bar{dz}}{d\xi} \right) \frac{d\bar{w}}{dz}$$

$$= \frac{d}{dz} \left(\Phi_1' \frac{dz}{d\xi} \right) \bar{f} \frac{d\bar{w}}{dz} + \frac{d}{dz} \left(\Phi_2' \frac{dz}{d\xi} \right) \frac{d\bar{w}'}{d\bar{z}} + \frac{d}{dz} \left(\Phi_3' \frac{dz}{d\xi} \right) \frac{d\bar{w}}{dz} \quad (22)$$

The following integrals must be calculated:

$$\int \frac{dw}{dz} \frac{dw}{d\zeta} dz; \quad \int \bar{w} \frac{d\bar{\zeta}}{dz} dz; \quad \int \bar{\zeta} \frac{d\bar{w}'}{d\bar{z}} d\bar{z}; \quad \int \bar{w} \frac{d\bar{w}'}{d\bar{z}} d\bar{z}; \quad \int \bar{\zeta} \frac{d\bar{w}}{d\bar{z}} d\bar{z}$$

From (19) follows

$$\zeta = \frac{z^2 + a^2}{z}; \quad w = \frac{R^2 - a^2}{z}; \quad \frac{dw}{dz} = - \frac{R^2 - a^2}{z^2}$$

$$w' = \frac{dw}{d\zeta} = - \frac{R^2 - a^2}{z^2 - a^2}; \quad \frac{dw'}{dz} = \frac{2z(R^2 - a^2)}{(z^2 - a^2)^2}$$

and elementary integration gives

$$\left. \begin{aligned} \int \frac{dw}{dz} \frac{dw}{d\zeta} dz &= (R^2 - a^2)^2 \int \frac{dz}{z^2 (z^2 - a^2)} = \frac{(R^2 - a^2)^2}{a^2} \left(\frac{1}{z} + \frac{1}{2a} \ln \frac{z-a}{z+a} \right) \\ \int \bar{w} \frac{d\bar{\zeta}}{dz} dz &= (R^2 - a^2) \int \frac{z^2 - a^2}{z^3} dz = (R^2 - a^2) \left(\ln z + \frac{a^2}{2z^2} \right) \\ \int \bar{\zeta} \frac{d\bar{w}}{d\bar{z}} d\bar{z} &= - (R^2 - a^2) \int \frac{z^2 + a^2}{z^3} dz = - (R^2 - a^2) \left(\ln z - \frac{a^2}{2z^2} \right) \\ \int \bar{w} \frac{d\bar{w}'}{d\bar{z}} d\bar{z} &= 2 (R^2 - a^2)^2 \int \frac{dz}{(z^2 - a^2)^2} = - \frac{(R^2 - a^2)^2}{a^2} \left(\frac{z}{z^2 - a^2} \right. \\ &\quad \left. + \frac{1}{2a} \ln \frac{z-a}{z+a} \right) \\ \int \bar{\zeta} \frac{d\bar{w}'}{d\bar{z}} d\bar{z} &= 2 (R^2 - a^2) \int \frac{z^2 + a^2}{(z^2 - a^2)^2} dz = - \frac{2 (R^2 - a^2) z}{z^2 - a^2} \end{aligned} \right\} (22a)$$

with (22) and (22a), (21) can be integrated, giving as particular integral of the nonhomogeneous equation (21)

$$\begin{aligned}
 4 W_2^* = & A \frac{w'^3}{3} \bar{\xi} + B w'^2 \bar{w} + C w' \frac{(R^2 - a^2)^2}{a^2} \left(\frac{1}{\bar{z}} + \frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} \right) \\
 & + \left(\frac{u^2 - 1}{4} \right)^2 \left(2 (\sigma + 2\kappa) \left[\sigma \left(\frac{w'^3}{3} \bar{\xi} + \frac{w'^2}{2} \bar{w} \right) \right. \right. \\
 & + (\sigma + 4) \left[\frac{w'^2}{2} \bar{w} + w' \frac{(R^2 - a^2)^2}{a^2} \left(\frac{1}{\bar{z}} + \frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} \right) \right] \\
 & + \sigma \left[\sigma w'^2 \frac{dw' dz}{dz d\xi} \frac{\bar{\xi}^2}{2} + (\sigma + 4) w' \frac{dw'}{dz} \frac{dz}{d\xi} (R^2 - a^2) \left(\ln \bar{z} + \frac{a^2}{2\bar{z}^2} \right) \right. \\
 & + w' \Phi'_3 \frac{dz}{d\xi} \bar{\xi} + \frac{\sigma}{2} \bar{w}' \frac{(R^2 - a^2)^2}{a^2} \left(\frac{1}{\bar{z}} + \frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} \right) + \bar{w}'^2 w (\sigma + 4) \left. \right] \\
 & - (\sigma + 4) \left[- \sigma w'^2 \frac{(R^2 - a^2) \bar{z}}{\bar{z}^2 - a^2} - (\sigma + 4) w' \frac{(R^2 - a^2)^2}{a^2} \left(\frac{\bar{z}}{\bar{z}^2 - a^2} \right. \right. \\
 & + \frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} \left. \right) + \Phi_3 \bar{w}' + \frac{\sigma}{6} w'^3 \bar{\xi} + \frac{\sigma + 4}{2} w'^2 \bar{w} \\
 & + (R^2 - a^2) \sigma w' \frac{dw' dz}{dz d\xi} \left(\ln \bar{z} - \frac{a^2}{2\bar{z}^2} \right) - (\sigma + 4) \frac{dw' dz}{dz d\xi} \frac{\bar{w}'^2}{2} \\
 & \left. \left. - \Phi'_3 \frac{dz}{d\xi} \bar{w} - (\sigma + 4) \bar{w}' \frac{(R^2 - a^2)^2}{a^2} \left(\frac{1}{\bar{z}} + \frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} \right) \right] \right) \quad (23)
 \end{aligned}$$

The two terms multiplied by $\ln \bar{z}$ counteract each other, as should be, as otherwise an impossible circulation could appear in this approximation. For large values of $|z| = R$, W_z^* behaves as $\frac{1}{r^8}$.

It is advisable to rearrange this expression:

$$\begin{aligned}
 W_2^* = & a_1 w'^3 \bar{\xi} + a_2 w'^2 \bar{w} + a_3 w' \left(\frac{1}{\bar{z}} + \frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} \right) \\
 & + a_4 \bar{w}' \left(\frac{1}{z} + \frac{1}{2a} \ln \frac{z - a}{z + a} \right) + a_5 w'^2 w'' \bar{\xi}^2 + a_6 w' w'' \left(\frac{1}{\bar{z}} \right)^2 \\
 & + a_7 \bar{w}'^2 w + a_8 w' \frac{d\Phi_3}{d\xi} \bar{\xi} + a_9 w'^2 \frac{\bar{z}}{\bar{z}^2 - a^2} + a_{10} \Phi_3 \bar{w}' \\
 & + a_{11} w'' \bar{w}'^2 + a_{12} \frac{d\Phi_3}{d\xi} \bar{w} + a_{13} w' \left(\frac{\bar{z}}{\bar{z}^2 - a^2} - \frac{1}{\bar{z}} \right) \quad (24)
 \end{aligned}$$

where the coefficients a_1 to a_{13} follow in simple manner from the coefficients of (23).

In order that W_2 comply with the boundary condition $W_2 = 0$ at the image-plane circle, suitable analytical functions of z and \bar{z} must be added to W_2^* , as they are homogeneous solutions of (21). For example, the term $w'^3 \bar{\xi}$ must be replaced by

$$w'^3 \left(\bar{\xi} - \frac{R^2}{z} - \frac{a^2 \bar{z}}{R^2} \right); \text{ a similar procedure must be used for the}$$

other terms. Certain terms are accompanied by new singularities in the outside domain, which must be compensated for by adding further suitable homogeneous solutions complying with the boundary conditions. Thus, the coefficient of a_3 must, in order to satisfy the boundary conditions (at the image-plane circle and at infinity) be substituted by

$$- a_3 (R^2 - a^2) \left(\frac{1}{z^2 - a^2} - \frac{\bar{z}^2}{R^4 - a^2 \bar{z}^2} \right) \left(\frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} + \frac{1}{\bar{z}} \right)$$

and in addition, for removing the singularities at $\bar{z} = \pm \frac{R^2}{a}$, the following function has to be added

$$a_3 (R^2 - a^2) \left(\frac{z}{z^2 - a^2} - \frac{R^2 \bar{z}}{R^4 - a^2 \bar{z}^2} \right) \left(\frac{1}{2a^2} \ln \frac{R^2 - a^2}{R^2 + a^2} + \frac{1}{R^2} \right)$$

Since $\frac{R^2 - a^2}{R^2 + a^2} = \epsilon$, a term $\sim \ln \epsilon$ occurs; the same is the case for a_4 . If one transforms all terms accordingly and makes a few simplifications, a solution of (21) is obtained which satisfies all boundary conditions and has no singularities in the outside domain. This solution is W_2 ; its imaginary part represents the third correction ψ_3 of the stream function:

$$\begin{aligned}
 W_2 = & a_1 w'^3 \left[\bar{z} - \frac{a^2}{z} + a^2 \left(\frac{1}{\bar{z}} - \frac{z}{R^2} \right) \right] + a_2 w'^2 (R^2 - a^2) \left(\frac{1}{\bar{z}} - \frac{z}{R^2} \right) \\
 & - a_3 (R^2 - a^2) \left[\left(\frac{1}{z^2 - a^2} - \frac{\bar{z}^2}{R^4 - a^2 \bar{z}^2} \right) \left(\frac{1}{\bar{z}} + \frac{1}{2a} \ln \frac{\bar{z} - a}{\bar{z} + a} \right) \right. \\
 & \left. - \left(\frac{z}{z^2 - a^2} - \frac{R^2 \bar{z}}{R^4 - a^2 \bar{z}^2} \right) \left(\frac{1}{2a^2} \ln \epsilon + \frac{1}{R^2} \right) \right] \\
 & - a_4 (R^2 - a^2) \left[\left(\frac{1}{\bar{z}^2 - a^2} - \frac{z^2}{R^4 - a^2 z^2} \right) \left(\frac{1}{z} + \frac{1}{2a} \ln \frac{z - a}{z + a} \right) \right. \\
 & \left. - \left(\frac{\bar{z}}{\bar{z}^2 - a^2} - \frac{R^2 z}{R^4 - a^2 z^2} \right) \left(\frac{1}{2a^2} \ln \epsilon + \frac{1}{R^2} \right) \right] \\
 & + a_5 w'^2 w'' \left[z^2 - \frac{R^4}{z^2} + a^4 \left(\frac{1}{\bar{z}^2} - \frac{z^2}{R^4} \right) \right] + a_6 w' w'' \left(\frac{1}{\bar{z}^2} - \frac{z^2}{R^4} \right) \\
 & + a_7 w'^2 (R^2 - a^2) \left(\frac{1}{z} - \frac{\bar{z}}{R^2} \right) + a_8 w' \frac{d\phi_3}{d\zeta} \left[\bar{z} - \frac{R^2}{z} + a^2 \left(\frac{1}{\bar{z}} - \frac{z}{R^2} \right) \right] \\
 & + a_9 w'^2 \left(\frac{\bar{z}}{\bar{z}^2 - a^2} - \frac{R^2 z}{R^4 - a^2 z^2} \right) - a_{10} \phi_3 (R^2 - a^2) \left(\frac{1}{\bar{z}^2 - a^2} - \frac{z^2}{R^4 - a^2 z^2} \right) \\
 & + a_{11} w'' (R^2 - a^2)^2 \left(\frac{1}{\bar{z}^2} - \frac{z^2}{R^4} \right) + a_{12} \frac{d\phi_3}{d\zeta} (R^2 - a^2) \left(\frac{1}{\bar{z}} - \frac{z}{R^2} \right) \\
 & + a_{13} w' \left[\frac{1}{\bar{z}} - \frac{z}{R^2} - \left(\frac{\bar{z}}{\bar{z}^2 - a^2} - \frac{R^2 z}{R^4 - a^2 z^2} \right) \right] \quad (25)
 \end{aligned}$$

with

$$\left. \begin{aligned}
 w' &= -\frac{R^2 - a^2}{z^2 - a^2}; \quad w'' = \frac{2z^3(R^2 - a^2)}{(z^2 - a^2)^3} \\
 \Phi_3 &= (R^2 - a^2) \left[-\frac{\sigma}{2(z^2 - a^2)^2} \left(\frac{a^2}{R^2} z + \frac{R^2}{z} \right) + \frac{\sigma + 4}{R^2} \frac{z}{z^2 - a^2} \right] \\
 \frac{d\Phi_3}{dz} &= \frac{(R^2 - a^2)^2 z^2}{R^2 (z^2 - a^2)^2} \left[\frac{2\sigma(a^2 z^2 + R^4)}{(z^2 - a^2)^2} \right. \\
 &\quad \left. - \frac{\sigma}{2(z^2 - a^2)} \left(a^2 - \frac{R^4}{z^2} \right) - 2 \frac{(\sigma + 4) z^2}{(z^2 - a^2)} + \sigma + 4 \right]
 \end{aligned} \right\} (25a)$$

as check it is easily seen that the residue of W_2 of z ,

$\bar{z} = \pm \frac{R^2}{a}$ disappears, hence the singularity at these points in (25) is only apparent; furthermore, in each term with $z \bar{z} = r^2$ the factor $R^2 - r^2$ can be removed, hence W_2 vanishes for $|z| = |\bar{z}| = R$; finally that the expression vanishes to a sufficient degree at infinity. Computing the normal derivative of ψ_3 on the ellipse from (25) and developing especially for $z = iR$, $\bar{z} = -iR$ (the end point of the small axis) in powers of ϵ , it is apparent that:

(1) The development of all terms starts with a term of the third order in ϵ ; in particular, at a_3 and a_4 through the logarithm, a term $\sim \epsilon^3 (\arccos \epsilon - 1)$ or in third order

$\sim \epsilon^3 \left(\frac{\pi}{2} - 1 \right)$ will occur that cannot be removed again by the terms equally proportional to ϵ^3 arising from ψ_1 and ψ_2 .

(2) In a_3 and a_4 an additive term proportional to $\epsilon^4 \ln \epsilon$ occurs likewise for which the same is true as in (1). Also this term must enter in the final formula for the velocity in the end point of the small axis. The order of magnitude of this term is still sufficiently large for a small but not too small ϵ .

compared with terms of the order ϵ^3 , thus it cannot be neglected for this order of approximation.

It has thus been proved with this special example that within the known terms, the Rayleigh approximation and the Prandtl-Busemann approximation give the same result.

Translated by J. Vanier
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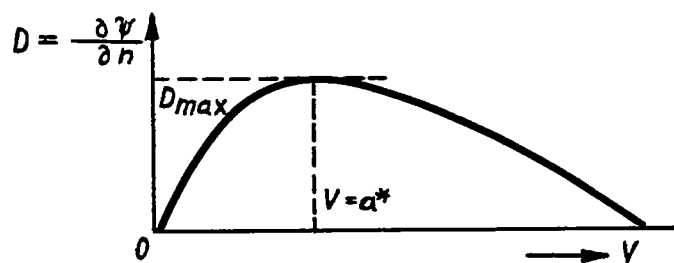


Figure 1.

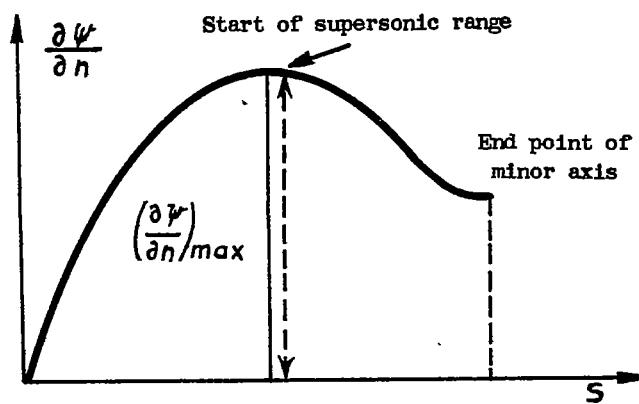


Figure 2.

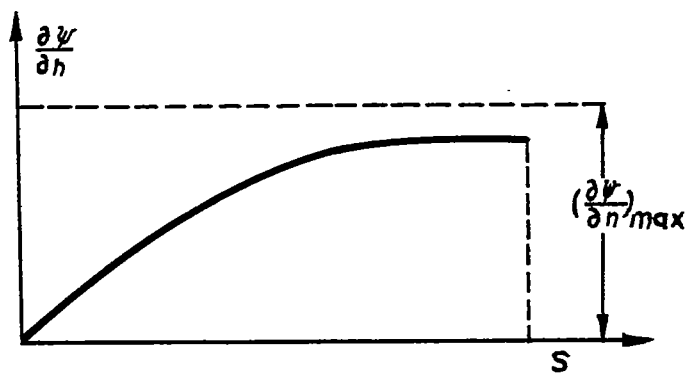


Figure 3.

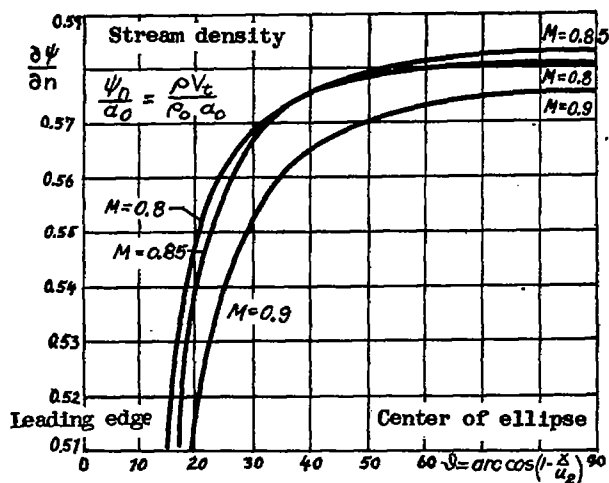


Figure 4.- Stream-density distribution along an ellipse ($d:l = 0.1$) for various Mach numbers.

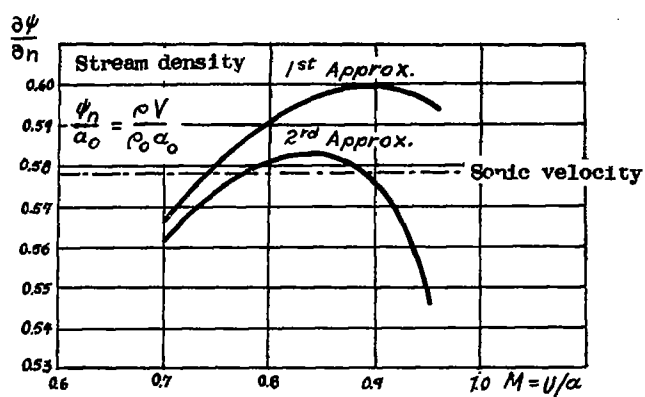


Figure 5.- Stream density at the end point of the minor axis of an ellipse ($d:l = 0.1$).

PART II. EFFECT OF COMPRESSIBILITY
IN AXIALLY SYMMETRICAL FLOW
AROUND AN ELLIPSOID

The differential equation for an axially symmetrical irrotational flow of a compressible gas is solved on the basis of the solution of Prandtl's linearized equation as first approximation for the case of axially symmetrical flow around an elongated ellipsoid; terms of higher than the second order in the interference velocities are neglected. The maximum velocities of ellipsoids of various thickness are calculated in dependence of the Mach number and the results are compared with those of Göthert (very slender ellipsoids; reference 1) and Lamla (sphere; reference 2).

SYMBOLS

x, r, φ	cylindrical coordinates in compressible flow
$\epsilon^* = d^*/t$	thickness ratio of body in the x -, r -plane of flow
U^*	free-stream velocity in direction of x -axis
$\sigma = x, \tau = r/\mu$	coordinates in the incompressible comparative flow
$\epsilon = d/t$	thickness ratio of body in the σ -, τ -plane
$U = \frac{\mu^2}{v} U^*$	free-stream velocity in the σ -, τ -plane
ρ, a	local density and sonic velocity of compressible flow
ρ_0, a_0	density and sonic velocity of air at rest
ρ_∞, a_∞	density and sonic velocity at U^*
$v = \rho_0/\rho_\infty$	density ratio
$M_\infty = U^*/a_\infty$	Mach number
$\mu = 1/\sqrt{1 - M_\infty^2}$	distortion factor

u, v	component of total velocity of compressible flow in x and r direction
$u_1 = u - U^*, v_1 = v$	disturbance velocity
Φ, ψ	flow potential and Stokes' stream function
Φ_1, ψ_1	potential and stream function of disturbance flow u_1, v_1
Φ_∞, ψ_∞	potential and stream function of the free stream
ξ, η	elliptic coordinates in the σ, τ -plane
k	scale factor of the elliptic coordinates
$\eta_0 = 1/\sqrt{1 - \epsilon^2}$	elliptic coordinate of the ellipsoid with thickness ratio ϵ
$C = \frac{-U/2}{\frac{\eta_0}{\eta_0^2 - 1} - \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1}}$	factor of the disturbance function
$\kappa = c_p/c_v$	ratio of specific heats

I. RESULTS AND NOTES FOR THE APPLICATION OF THE PRANDTL RULE TO ELLIPSOIDS OF REVOLUTION

In order to describe the flow around an ellipsoid of revolution cylindrical coordinates x, r, φ are introduced. The x -axis is placed in the free-stream direction, r is the distance from the x -axis, and φ the angle of rotation. The flow yields the same pattern for each $\varphi = \text{const}$; thus the differential equation of the flow is independent of φ

$$\left. \begin{aligned} \Phi_{xx} \left(1 - \frac{u^2}{a^2}\right) + \Phi_{rr} \left(1 - \frac{v^2}{a^2}\right) - \frac{2uv}{a^2} \Phi_{xr} + \frac{\Phi_r}{r} &= 0 \\ \psi_{xx} \left(1 - \frac{u^2}{a^2}\right) + \psi_{rr} \left(1 - \frac{v^2}{a^2}\right) - \frac{2uv}{a^2} \psi_{xr} - \frac{\psi_r}{r} &= 0 \end{aligned} \right\} \quad (1)$$

Φ is the potential defined by irrotationality, u and v are the velocity components in the x and r directions, and ψ is the Stokes stream function defined by the equation of continuity. If one substitutes the disturbance velocities

$$u_1 = u - U^*, \quad v_1 = v$$

for ψ instead of the total velocities in the equation, where U^* is the free stream in compressible flow, and expresses the local sonic velocity a by the velocities and the sonic velocity a_∞ of the flow, one obtains similar to Busemann (reference 3) the disturbance stream function $\psi_1 = \psi - \psi_\infty$ for the two-dimensional case

$$\begin{aligned} \psi_{1xx} \left(1 - M_\infty^2\right) + \psi_{1rr} - \frac{\psi_{1r}}{r} = & - \frac{v U^*}{a_\infty^2} \left[\mu^2 (\kappa + 1) \psi_{1xx} \frac{\psi_{1r}}{r} \right. \\ & + \mu^2 (\kappa - 1) \psi_{1rr} \frac{\psi_{1r}}{r} \\ & - 2 \psi_{1xr} \frac{\psi_{1x}}{r} + \frac{\psi_{1x}^2}{r^2} \\ & \left. - (\kappa - 1) \mu^2 \frac{\psi_{1r}^2}{r^2} \right] \quad (2) \end{aligned}$$

$M_\infty = \frac{U^*}{a_\infty}$ is the free-stream Mach number, $v = \frac{\rho_0}{\rho_\infty}$ the density ratio at rest, ($\rho = \rho_0$), and for free stream ($\rho = \rho_\infty$). If ψ_∞ is the stream function of the free stream

$$U^* = -v \frac{\psi_{\infty r}}{r} \quad (3)$$

omitting the second power terms in the derivatives of the stream function on the right-hand side, there remains the linearized differential equation of the axially symmetrical flow whose solution

corresponds to the Prandtl approximation. By distorting the x, r plane with the factor $\mu = 1/\sqrt{1 - M_\infty^2}$

$$x = \sigma \quad r = \mu \tau \quad (4)$$

the linearized equation gives the differential equation of the incompressible ellipsoid flow, the solution of which is immediately indicated. Substituting the solution $\psi_1^{(1)}$ of the homogeneous equation in (2) on the right-hand side, the solution of the now inhomogeneous equation will give a solution $\psi_1^{(2)}$ and

$$\psi_1 = \psi_1^{(1)} + \psi_1^{(2)} \quad (5)$$

represents a higher approximation for the solution of the compressible flow differential equation. The solution of the linearized differential equation, that is, the first approximation for the compressible flow, is in elliptical coordinates (Lamb, reference 4)

$$\sigma = k\xi\eta \quad \tau = k\sqrt{1 - \xi^2} \sqrt{\eta^2 - 1} \quad (6)$$

for an ellipsoid of thickness ratio ϵ^* and free-stream velocity

U^* in the distorted σ, τ -plane $\epsilon = \frac{\epsilon^*}{\mu}$

$$U = -\frac{\psi_{\sigma\tau}}{\tau} = \frac{\mu^2}{U} U^* \psi_1^{(1)} = C k^2 (1 - \xi^2) (\eta^2 - 1) \left(\frac{1}{2} \ln \frac{\eta + 1}{\eta - 1} - \frac{\eta}{\eta^2 - 1} \right)$$

with

$$\dot{C} = \frac{-U/2}{\frac{\eta_0}{\eta_0^2 - 1} - \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1}} \quad (7)$$

For $\eta = \eta_0 = \frac{1}{\sqrt{1 - \epsilon^2}}$ the coordinates σ, τ represent the ellipsoid of thickness ratio ϵ and the first approximation of the velocity u_1 follows from the solution $\psi_1^{(1)}$

$$u_{1P}^{(1)} = -\mu^2 \psi \frac{\psi_{1r}^{(1)}}{r} = -\psi \frac{\psi_1^{(1)} r}{r} = -\psi \frac{\psi_{1\eta}^{(1)}}{\eta} \quad (8)$$

The higher terms in the derivatives of the stream function are neglected. If one includes the square terms for the calculation of the second approximation, the first approximation

$\psi_1^{(1)}$ gives an additional term $\Delta u_1^{(1)}$ in the velocity u_1 of higher order

$$u_1^{(1)} = u_{1P}^{(1)} + \Delta u_1^{(1)} \quad (9)$$

Substituting (7) in (8) gives the Prandtl approximation for the maximum speed ($\xi = 0$) at the contour of the ellipsoid ($\eta = \eta_0$)

$$\left[u_{1P}^{(1)} \right]_{\max} = U^* \mu^2 \left\{ \frac{\frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} - \frac{1}{\eta_0}}{\frac{\eta_0}{\eta_0^2 - 1} - \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1}} \right\} \quad (10)$$

with

$$\eta_0 = \frac{1}{\sqrt{1 - (\epsilon^*/\mu)^2}} \approx 1 + \frac{(\epsilon^*/\mu)^2}{2}$$

Neglecting the terms of higher order in ϵ for thin ellipsoids ($\epsilon^2 \ll 1$) gives (asymptotic values for $\eta_0 \rightarrow 1$, first approximation)

$$\left[\frac{u_{1P}^{(1)}}{U^*} \right]_{\max} \approx \epsilon^{*2} \left(\ln \frac{2\mu}{\epsilon^*} - 1 \right) \quad (11)$$

With the same omission in incompressible flow

$$\left(\frac{u_1 \text{ incomp.}}{U^*} \right)_{\max} \approx \epsilon^{*2} \left(\ln \frac{2}{\epsilon^*} - 1 \right)$$

¹This result of the first approximation can be entered at once according to the Prandtl rule, when Götthert's form (1) is applied: determine the interference velocity in the incompressible flow at a contour distorted with $1/\mu$ and multiplied speed by μ^2 .

The ratio of the compressible to the incompressible maximum velocity for an ellipsoid of small thickness ratio ϵ thus is

$$\left(\frac{u_1 \text{ compr.}}{u_1 \text{ incomp.}} \right)_{\max} \approx \frac{\ln \frac{2\mu}{\epsilon} - 1}{\ln \frac{2}{\epsilon} - 1} = 1 + \frac{\ln \mu}{\ln \frac{2}{\epsilon} - 1}^2 \quad (12)$$

Consequently, the ratio of the velocities for thin ellipsoids is not unity in first approximation. Rather an additional term proportional to $\ln \mu$ is obtained, which at higher Mach numbers can be quite considerable even for small thicknesses. The value of unity is attained in the limiting process to vanishing thickness at fixed Mach number. Considering that the first approximation of the potential $\psi_1^{(1)}$ by (7) is, like the velocity in (11), proportional to ϵ^2 , it is apparent from (2) that the additive stream function $\psi_1^{(2)}$ can contain only terms proportional to ϵ^4 . Thus (11) certainly contains terms all proportional to ϵ^2 . For the second approximation of the maximum velocity at the ellipsoid with

$$\frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} = Q_0$$

as abbreviation, one obtains the following result

$$\left. \begin{aligned} u_{1\max} &= u_1^{(1)}_{\max} + u_1^{(2)}_{\max} \\ \text{with} \quad u_1^{(1)}_{\max} &= \left[u_{1p}^{(1)} \right]_{\max} + \Delta u_1^{(1)}_{\max} \end{aligned} \right\} \quad (13)$$

Göthert obtained a similar result $\ln \mu / \ln \frac{1}{\epsilon}$, as is apparent from the above when $(\ln 2) - 1$ is neglected with respect to $\ln \epsilon$. But such an omission is admissible only for extremely small ϵ , because $\ln \epsilon = -2.996$ is itself no longer great with respect to $(\ln 2) - 1 = 0.307$, so that errors of 10 percent result.

where

$$\left[u_{1P}^{(1)} \right]_{\max} = \mu^2 U^* \frac{\eta_0^2 - 1}{\eta_0 - (\eta_0^2 - 1) Q_0} \left(Q_0 - \frac{1}{\eta_0} \right)$$

denotes Prandtl's first approximation, while

$$\Delta u_1^{(1)}_{\max} = \mu^4 M_\infty^2 U^* \left[1 + \frac{\mu^2}{2} (1 + \kappa M_\infty^2) \right] \left[\frac{(\eta_0^2 - 1) (Q_0 - \frac{1}{\eta_0})}{\eta_0 - (\eta_0^2 - 1) Q_0} \right]^2$$

and

$$u_1^{(2)}_{\max} = \frac{1}{4} \mu^4 M_\infty^2 U^* \left[\frac{\eta_0^2 - 1}{\eta_0 - (\eta_0^2 - 1) Q_0} \right]^2 \left\{ \left[1 + \mu^2 (1 + M_\infty^2) \right] \left[\frac{\psi_{I\eta}(2)}{\eta} \right]_{\eta=\eta_0} + \left[\frac{\psi_{II\eta}(2)}{\eta} \right]_{\eta=\eta_0} \right\} \quad (14)$$

represent the additional portions of the second approximations.

The expressions $\left[\frac{1}{\eta} \psi_{I\eta}(2) \right]_{\eta=\eta_0}$ and $\left[\frac{1}{\eta} \psi_{II\eta}(2) \right]_{\eta=\eta_0}$ are solely

dependent upon $\eta_0 = 1/\sqrt{1 - (\epsilon^*/\mu)^2}$, that is, on the ratio ϵ^*/μ . They are shown in figure 1 for small $\eta_0 \geq 1$ and therefore immediately available for every practical example: η_0 is the larger the thicker the ellipsoid and the smaller the Mach number. Even at $\epsilon^* = 0.5$ and $M_\infty = 0.3$ η_0 is not larger than 1.138, hence still a number close to $\eta_0 = 1$.

Figure 2 shows the maximum velocity increments of the second approximation $(u_1/U^*)_{\max} = \left(\frac{u - U^*}{U^*} \right)_{\max}$ on ellipsoids of various thickness ratios ϵ^* plotted against the Mach number. The range of validity of the curves is bounded by the curve of the critical Mach number, on which the local velocity exactly reaches sonic velocity. The heavy curves of the second approximation $(u_1/U^*)_{\max}$ are compared with the first approximation $[u_{1P}^{(1)}/U^*]_{\max}$, which

follows from the first approximation of the stream function $\psi_1^{(1)}$ when neglecting the square terms in the derivatives of $\psi_1^{(1)}$.

This first approximation is represented by thin curves. To illustrate the magnitude of the two additional terms of higher order the total contribution by the linearized differential equation

$[u_1^{(1)}/U^*]_{\max}$ is included in dashed lines. It is seen that the additive term $\Delta u_1^{(1)}$ contributes entirely too much and is reduced again to the greater part by the term of the second approximation $u_1^{(2)}$.

In order to judge the quality of the second approximation, the maximum incremental velocities for the extreme case of the sphere $d^*/t = \epsilon^* = 1$ are plotted in figure 3 and compared with Lamla's values for the sphere at several Mach numbers. Since Lamla referred the Mach number to the critical velocity of sound a^* , the Mach numbers $\beta = U^*/a^*$ were converted to the sonic velocity a_∞ at free-stream velocity U^* by the formula

$$M_\infty^2 = \frac{U^{*2}}{a_\infty^2} = \frac{2\beta^2}{(\kappa + 1) - (\kappa - 1)\beta^2}$$

It is seen that Lamla's values computed in fourth approximation by the Rayleigh method do not differ very much from the second approximation computed here; hence it may be concluded that this second approximation is surely sufficient for all practically encountered ellipsoids with thickness ratio up to about $\epsilon^* = 0.5$.

Figure 4 represents the conditions for the incremental velocities in compressible and incompressible flow around bodies of the same contour plotted against the Mach number for various thickness ratios of the ellipsoid in first and second approximation. Lamla's values are included for comparison in dash-dotted lines. Figure 6 contains the corresponding conditions for the total velocities $u = u_1 + U^*$ in second approximation. Figures 4 and 6 represent the final result of the higher approximation which goes beyond the Prandtl rule. The compressible disturbance and total velocities of the second approximations are directly obtainable, if the incompressible values are taken from figure 8.

In the calculation of the second approximation of the stream function and its derivatives the terms of the solution $\psi_{1\eta}^{(2)}$

were represented by infinite series which, for $\eta_0 \rightarrow 1$, tend toward logarithmic limiting curves. If one replaces these series for thin ellipsoids by their limiting curves and considers the terms up to the order ϵ^4 one obtains for the incremental velocity the asymptotic formula (second approximation)

$$\begin{aligned} \left(\frac{u_1}{U^*} \right)_{\max} &\approx \epsilon^2 \left(\ln \frac{2\mu}{\epsilon^*} - 1 \right) \\ &+ \frac{\epsilon^4}{2\mu^2} \left[2 \ln^2 \frac{2\mu}{\epsilon^*} + (2\mu^2 - 3) \ln \frac{2\mu}{\epsilon^*} - \frac{7}{2} \mu^2 + 3 \right] \\ &+ \frac{\epsilon^4 (\mu^2 - 1)}{4} (1 + \kappa M_\infty^2) \end{aligned} \quad (15)$$

The asymptotic values for the partial solutions $\psi_{I\eta}^{(2)}$ and $\psi_{II\eta}^{(2)}$ employed here are indicated in figure 1. For the incompressible case for $\mu = 1$ the asymptotic formula is true

$$\begin{aligned} \left(\frac{u_1 \text{ incomp.}}{U^*} \right)_{\max} &\approx \epsilon^2 \left(\ln \frac{2}{\epsilon^*} - 1 \right) \\ &+ \frac{\epsilon^4}{2} \left[2 \ln^2 \frac{2}{\epsilon^*} - \ln \frac{2}{\epsilon^*} - \frac{1}{2} \right] \end{aligned} \quad (16)$$

Thus the ratio of interference velocities is

$$\begin{aligned} \left(\frac{u_1 \text{ compr.}}{u_1 \text{ incomp.}} \right)_{\max} &\approx 1 + \frac{\ln \mu}{\ln \frac{2}{\epsilon^*} - 1} + \frac{\epsilon^2}{2 \left(\ln \frac{2}{\epsilon^*} - 1 \right)} \left[\frac{2}{\mu^2} \ln^2 \frac{2\mu}{\epsilon^*} - 2 \ln^2 \frac{2}{\epsilon^*} \right. \\ &+ \frac{2\mu^2 - 3}{\mu^2} \ln \frac{2\mu}{\epsilon^*} + \ln \frac{2}{\epsilon^*} - 3 + \frac{3}{\mu^2} + \frac{\mu^2 - 1}{2} (1 + \kappa M_\infty^2) \\ &\left. - \frac{\ln \mu}{\ln \frac{2}{\epsilon^*} - 1} \left(2 \ln^2 \frac{2}{\epsilon^*} - \ln \frac{2}{\epsilon^*} - \frac{1}{2} \right) \right], \end{aligned} \quad (17)$$

For the total velocity $u = u_1 + U^*$ the ratio is

$$\left(\frac{u_{\text{compr.}}}{u_{\text{incompr.}}} \right)_{\text{max}} \approx 1 + \epsilon^{*2} \ln \mu + \frac{\epsilon^{*4}}{2\mu^2} (\mu^2 - 1) \\ \left[-2 \ln \frac{2\mu}{\epsilon^*} \ln \frac{2}{\epsilon^*} + 3 \left(\ln \frac{2}{\epsilon^*} - 1 \right) + \frac{\mu^2}{2} (1 + \kappa M_\infty^2) \right] \\ + \frac{\epsilon^{*4}}{2\mu^2} \ln \mu \left(2 \ln \frac{2\mu}{\epsilon^*} + 4\mu^2 - 3 \right) \quad (18)$$

The small crosses indicate the asymptotic values of the second approximation for $\epsilon^* = 0.2$ in figure 2; one can see that they are still a little below the first approximation at small Mach numbers, between the first and second approximation at medium M_∞ and then above the second approximation. Then they finally approach the second approximation asymptotically at $M_\infty \rightarrow 1$, since for $\mu \rightarrow \infty$ in $\eta_0 = \frac{1}{\sqrt{1 - (\epsilon^*/\mu)^2}}$ and thus $\eta_0 \rightarrow 1$.

Figure 5 contains, aside from the exact second approximation, the values of the second approximation computed by the asymptotic formula (1) for $\epsilon^* = 0.2$. Thus the two second approximations do not differ very much for small thicknesses. If one neglects the terms of higher order in (17) one obtains the asymptotic values (11) of the Prandtl approximation corresponding to the exact values of the first approximation. One can see from figure 5 that the exact values are too small, the asymptotic values too high, hence their average is close to the second approximation. The dotted curve in figure 6 represents the asymptotic values of the first approximation for $\epsilon^* = 0.2$ corresponding to (17) with neglect of the terms with ϵ^{*4} ; the dashed curve indicates the exact first solution.

II. OUTLINE OF THE THEORETICAL DERIVATION OF THE RESULTS

(a) The Differential Equation for Potential and Stream Function

The continuity equation of a compressible medium for axially symmetrical flow with local density ρ and the local velocity vector \underline{w} is

$$\text{div} (\rho \underline{w}) = 0 \quad (19)$$

Expressed in cylindrical coordinates x, r, ϕ with $\underline{w} = (u, v, w)$ it becomes

$$\frac{1}{r} \left(\frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial r} - \frac{\partial \rho w}{\partial \phi} \right) = 0 \quad (20)$$

In the axially symmetric case the flow in every plane $\phi = \text{constant}$ is the same, hence only the flow in one x, r plane needs to be analyzed. In this plane ($w = 0$)

$$\frac{\partial}{\partial x} (\rho r u) + \frac{\partial}{\partial r} (\rho r v) = 0 \quad (21)$$

Thus there exists a Stokes stream function ψ with the property

$$u = -\frac{1}{r} \frac{\rho_0}{\rho} \psi_r, \quad v = \frac{1}{r} \frac{\rho_0}{\rho} \psi_x$$

If the flow is irrotational, there exists in addition a flow potential Φ , hence

$$u = \Phi_x = -\frac{1}{r} \frac{\rho_0}{\rho} \psi_r, \quad v = \Phi_r = \frac{1}{r} \frac{\rho_0}{\rho} \psi_x \quad (22)$$

Thus from the disappearance of the rotation, $\text{rot} \underline{w} = 0$, there follows for the stream function

$$\frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\rho_0}{\rho} \psi_x \right) + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\rho_0}{\rho} \psi_r \right) = 0 \quad (23)$$

Moreover, by Bernoulli's equation

$$\int \frac{dp}{\rho} + \frac{1}{2} q^2 = b = \text{Constant}$$

with

$$q^2 = u^2 + v^2 = \left(\frac{\rho_0}{\rho}\right)^2 \frac{1}{r^2} (\psi_r^2 + \psi_x^2)$$

and by definition of the sonic velocity $a^2 = dp/d\rho$ the relation follows

$$\frac{d\rho}{\rho} = d \ln \rho = - \frac{1}{2a^2} dq^2 \quad (24)$$

Substituting (24) in the basic equation (23) gives

$$\frac{\partial}{\partial x} \left(\frac{\psi_x}{r} \right) + \frac{\partial}{\partial r} \left(\frac{\psi_r}{r} \right) + \frac{1}{r} \left(\psi_x \frac{\partial \ln \rho}{\partial x} + \psi_r \frac{\partial \ln \rho}{\partial r} \right) \quad (25)$$

with

$$\begin{aligned} \frac{\partial \ln \rho}{\partial x} &= - \frac{1}{2a^2} \frac{\partial}{\partial x} \left[\left(\frac{\rho_0}{\rho} \right) \frac{2\psi_x^2 + \psi_r^2}{r^2} \right] = - \frac{1}{a^2} \left(\frac{\rho_0}{\rho} \right)^2 \frac{\psi_x \psi_{xx} + \psi_r \psi_{rx}}{r^2} + \frac{\partial \ln \rho}{\partial x} \frac{q^2}{a^2} \\ \frac{\partial \ln \rho}{\partial r} &= - \frac{1}{2a^2} \frac{\partial}{\partial r} \left[\left(\frac{\rho_0}{\rho} \right)^2 \frac{\psi_x^2 + \psi_r^2}{r^2} \right] \\ &= - \frac{1}{a^2} \left(\frac{\rho_0}{\rho} \right)^2 \left(\frac{\psi_x \psi_{xr} + \psi_r \psi_{rr}}{r^2} - \frac{\psi_x^2 + \psi_r^2}{r^3} \right) + \frac{\partial \ln \rho}{\partial r} \frac{q^2}{a^2} \end{aligned}$$

or better

$$\left. \begin{aligned} \frac{\partial \ln \rho}{\partial x} \left(1 - \frac{q^2}{a^2} \right) &= - \frac{1}{a^2} \frac{\rho_0}{\rho} \left(v \frac{\psi_{xx}}{r} - u \frac{\psi_{rx}}{r} \right) \\ \frac{\partial \ln \rho}{\partial r} \left(1 - \frac{q^2}{a^2} \right) &= - \frac{1}{a^2} \frac{\rho_0}{\rho} \left(v \frac{\psi_{xr}}{r} - u \frac{\psi_{rr}}{r} \right) + \frac{q^2}{a^2} \frac{1}{r} \end{aligned} \right\} \quad (26)$$

The basic equation thus is

$$\begin{aligned} \left(1 - \frac{q^2}{a^2} \right) (\psi_{xx} + \psi_{rr} - \frac{\psi}{r}) &= - \frac{1}{a^2} \left[v (\psi_{xx} - u \psi_{rx}) - u (v \psi_{xr} - u \psi_{rr}) \right] \\ &\quad + \frac{q^2}{a^2} \frac{\psi}{r} \end{aligned}$$

or

$$\psi_{xx} \left(1 - \frac{u^2}{a^2} \right) + \psi_{rr} \left(1 - \frac{v^2}{a^2} \right) - 2 \frac{uv}{a^2} \psi_{xr} - \frac{\psi}{r} = 0 \quad (27)$$

The differential equation for the potential is obtained in the same way. From the equation of continuity (21) and the Bernoulli equation (24) there follows

$$\frac{\partial}{\partial x} (r \phi_x) + \frac{\partial}{\partial r} (r \phi_r) + r \phi_x \frac{\partial \ln \rho}{\partial x} + r \phi_r \frac{\partial \ln \rho}{\partial r} = 0$$

with

$$\frac{\partial \ln \rho}{\partial x} = - \frac{1}{2a^2} \frac{\partial q^2}{\partial x} = - \frac{1}{a^2} (\phi_r \phi_{rx} + \phi_x \phi_{xx})$$

$$\frac{\partial \ln \rho}{\partial r} = - \frac{1}{2a^2} \frac{\partial q^2}{\partial r} = - \frac{1}{a^2} (\phi_r \phi_{rr} + \phi_x \phi_{rx})$$

Hence the differential equation for the potential is

$$\begin{aligned} \Phi_{xx} + \Phi_{rr} + \frac{\Phi_r}{r} - \frac{1}{a^2} (\Phi_x \Phi_r \Phi_{rx} + \Phi_x^2 \Phi_{xx}) \\ - \frac{1}{a^2} (\Phi_r^2 \Phi_{rr} + \Phi_r \Phi_x \Phi_{rx}) = 0 \end{aligned}$$

or, shorter

$$\Phi_{xx} \left(1 - \frac{u^2}{a^2}\right) + \Phi_{rr} \left(1 - \frac{v^2}{a^2}\right) - 2 \frac{uv}{a^2} \Phi_{xr} + \frac{\Phi_r}{r} = 0 \quad (28)$$

(b) Introduction of the Disturbance Velocities and Transformation
to Elliptic Coordinates in the Plane Distorted

According to Prandtl

From the adiabatic equation $p/p_0 = (\rho/\rho_0)^\kappa$, $\kappa = 1.405$ (air) and the definition of the sonic velocity $a^2 = dp/d\rho$ there follows when the pressure is expressed by q^2 according to the Bernoulli equation, $a^2 = a_0^2 - \frac{\kappa - 1}{2} q^2$, a_0 = sonic velocity of the gas at rest.

By means of this relation the density can be expressed by the disturbance velocities

$$u_1 = u - U^*; \quad v_1 = v$$

If a_∞ is the velocity of sound at free-stream velocity, $q = U^*$ and ρ_∞ the corresponding density, there follows with $\frac{\rho_0}{\rho_\infty} = v$

$$\begin{aligned} \frac{\rho}{\rho_0} &= \left(\frac{a^2}{a_0^2}\right)^{\frac{1}{\kappa-1}} = \frac{\rho_\infty}{\rho_0} \left(\frac{a^2}{a_\infty^2}\right)^{\frac{1}{\kappa-1}} \\ &= \frac{1}{v} \left[1 - (\kappa - 1) \frac{U^* u_1}{a_\infty^2} - \frac{1}{2} \frac{\kappa - 1}{a_\infty^2} (u_1^2 + v_1^2) \right]^{\frac{1}{\kappa-1}} \end{aligned}$$

or

$$\frac{\rho_0}{\rho} \approx v \left\{ 1 + \frac{U^* u_1}{a_\infty^2} + \frac{1}{2a_\infty^2} \left[(1 + \kappa M_\infty^2) u_1^2 + v_1^2 \right] \right\} \quad (29)$$

with $\frac{U^{*2}}{a_\infty^2} = M_\infty^2$

Splitting off the contribution of the free-stream velocity in the stream function there remains

$$\psi = \psi_\infty + \psi_1 \text{ with } \psi_\infty = -\frac{1}{2v} U^* r^2$$

because

$$U^* = -\frac{1}{r} \frac{\rho_0}{\rho_\infty} \psi_{\infty r} = -v \frac{\psi_{\infty r}}{r}$$

Now the disturbance velocities can be expressed by the disturbance stream function ψ_1 .

From

$$U^* + u_1 = -\frac{\rho_0}{\rho} \frac{1}{r} (\psi_{\infty r} + \psi_{1r}) \quad \text{or} \quad u_1 = U^* \left(\frac{1}{v} \frac{\rho_0}{\rho} - 1 \right)$$

$$-\frac{\rho_0}{\rho} \frac{\psi_{1r}}{r} \approx u_1 M_\infty^2 + \frac{U^*}{2a_\infty^2} \left[(1 - \kappa M_\infty^2) u_1^2 + v_1^2 \right] - \frac{\rho_0}{\rho} \frac{\psi_{1r}}{r}$$

follows with $\mu^2 = \frac{1}{1 - M_\infty^2}$

$$u_1 \approx v \mu^2 \left(1 + \frac{U^* u_1}{a_\infty^2} \right) \frac{\psi_{1r}}{r} + \frac{U^* \mu^2}{2a_\infty^2} \left[(1 - \kappa M_\infty^2) u_1^2 + v_1^2 \right]$$

$$v_1 \approx \frac{\rho_\infty}{\rho} \frac{\psi_{1x}}{r} = v \left(1 + \frac{U^* u_1}{a_\infty^2} \right) \frac{\psi_{1x}}{r} \quad (30)$$

Hence in first approximation

$$u_1 \approx -u_\infty^2 \frac{\psi_{1r}}{r} \quad v_1 \approx v \frac{\psi_{1x}}{r} \quad (30)$$

If one substitutes this approximation (30) in the terms of higher order one obtains as the second approximation

$$\begin{aligned} u_1 \approx -u_\infty^2 \frac{\psi_{1r}}{r} \left(1 - \frac{u_\infty^2 U^* \psi_{1r}}{a_\infty^2 r} \right) \\ + \frac{U^* u_\infty^2 v^2}{2a_\infty^2} \left[\left(1 + \kappa M_\infty^2 \right) \mu^4 \frac{\psi_{1r}^2}{r^2} + \frac{\psi_{1x}^2}{r^2} \right] \quad (31) \\ v_1 \approx v \frac{\psi_{1x}}{r} \left(1 - \frac{u_\infty^2 U^* \psi_{1r}}{a_\infty^2 r} \right) \end{aligned}$$

Substituting these disturbance velocities in the equation of the stream function (27), and neglecting all terms of higher than the second order one obtains, with

$$\psi_r = -\frac{1}{v} U^* r + \psi_{1r} \quad \psi_{rr} = -\frac{U^*}{v} + \psi_{1rr}$$

and

$$a^2 \approx a_\infty^2 - (\kappa - 1) U^* u_1$$

$$\begin{aligned} \psi_{1xx} \left[a_\infty^2 - (\kappa - 1) U^* u_1 - U^{*2} - 2U^* u_1 \right] \\ + \psi_{1rr} \left[a_\infty^2 - (\kappa - 1) U^* u_1 \right] - 2U^* v_1 \psi_{1xr} \\ - \frac{\psi_{1r}}{r} \left[a_\infty^2 - (\kappa - 1) U^* u_1 \right] + \frac{U^*}{v} v_1^2 = 0 \end{aligned}$$

or

$$\begin{aligned} \psi_{1xx} \left[1 - M_{\infty}^2 + (\kappa + 1) \frac{U^*}{a_{\infty}^2} v \mu^2 \frac{\psi_{1r}}{r} \right] + \psi_{1rr} \left[1 + (\kappa - 1) \frac{U^*}{a_{\infty}^2} v \mu^2 \frac{\psi_{1r}}{r} \right] \\ - 2v \frac{U^*}{a_{\infty}^2} \psi_{1xr} \frac{\psi_{1x}}{r} + v \frac{U^*}{a_{\infty}^2} \left(\frac{\psi_{1x}}{r} \right)^2 \\ - \frac{\psi_{1r}}{r} \left[1 + (\kappa - 1) \frac{U^*}{a_{\infty}^2} v \mu^2 \frac{\psi_{1r}}{r} \right] = 0 \end{aligned}$$

Thus under consideration of the terms up to the second order the differential equation of the disturbance stream function is

$$\begin{aligned} \psi_{1xx} (1 - M_{\infty}^2) + \psi_{1rr} - \frac{\psi_{1r}}{r} = - \frac{v U^*}{a_{\infty}^2} \left[\mu^2 (\kappa + 1) \psi_{1xx} \frac{\psi_{1r}}{r} \right. \\ \left. + \mu^2 (\kappa - 1) \psi_{1rr} \frac{\psi_{1r}}{r} - 2 \psi_{1xr} \frac{\psi_{1x}}{r} + \frac{\psi_{1x}^2}{r^2} - (\kappa - 1) \mu^2 \frac{\psi_{1r}^2}{r^2} \right] \end{aligned} \quad (32)$$

Distorting the coordinates according to Prandtl $x = \sigma$ $r = \mu \tau$, one obtains, because of

$$\psi_{1r} = \frac{1}{\mu} \psi_{\tau}; \quad \psi_{1rr} = \frac{1}{\mu^2} \psi_{\tau\tau}; \quad \psi_{1x} = \psi_{\sigma};$$

$$\begin{aligned} \psi_{\sigma\sigma} + \psi_{\tau\tau} - \frac{\psi_{\tau}}{\tau} = - v \frac{U^*}{a_{\infty}^2 \tau} \left[\mu^2 (\kappa - 1) \psi_{\sigma\sigma} \psi_{\tau} + (\kappa - 1) \psi_{\tau\tau} \psi_{\tau} \right. \\ \left. - 2 \psi_{\sigma\tau} \psi_{\sigma} + \frac{\psi_{\sigma}^2}{\tau} - (\kappa - 1) \frac{\psi_{\tau}^2}{\tau} \right] \end{aligned} \quad (33)$$

If $\psi^{(1)}$ is a solution of the homogeneous differential equation, that is,

$$\psi_{\sigma\sigma}^{(1)} + \psi_{\tau\tau}^{(1)} - \frac{\psi_{\tau}^{(1)}}{\tau} = 0$$

the insertion of this solution as first approximation in the right-hand side of (33) gives, with

$$\psi = \psi^{(1)} + \psi^{(2)}$$

for the correction to Prandtl's solution the inhomogeneous differential equation

$$\psi_{\sigma\sigma}^{(2)} + \psi_{\tau\tau}^{(2)} - \frac{\psi_{\tau}^{(2)}}{\tau} = - \frac{vU^*}{a_{\infty}^2 \tau} \left\{ \psi_{\sigma\sigma}^{(1)} \psi_{\tau}^{(1)} \left[\mu^2 (\kappa + 1) - (\kappa - 1) \right] + \psi_{\sigma}^{(1)} \left[\frac{\psi_{\sigma}^{(1)}}{\tau} - 2\psi_{\sigma\tau}^{(1)} \right] \right\} \quad (34)$$

Since the solution (7) of the homogeneous equation is given in elliptic coordinates, (34) must be transformed to the new coordinates. The simplest way is achieved by not using the transformation (6) but rather by mapping the σ, τ plane at first on an auxiliary plane ξ, θ by means of the analytic function

$$\sigma + i\tau = k \cosh (\xi + i\theta) \quad (35)$$

The transition to elliptic coordinates is then given by

$$\eta = \cosh \xi \quad \zeta = \cos \theta \quad (36)$$

The existence of the Cauchy-Riemann differential equations for the mapping (35) facilitates the transformation considerably and (see also appendix, section a) the transformation by means of (36) involves merely rewriting of the result with the new notation. The differential equation (34) attains then the form

$$\begin{aligned}
& (\eta^2 - 1) \psi_{\eta\eta}^{(2)} + (1 - \xi^2) \psi_{\xi\xi}^{(2)} \\
& = -\frac{vU^*}{a_\infty^2} 2 \frac{C^2 k^2}{(\eta^2 - \xi^2)^2} (1 - \xi^2) \left\{ \left[\mu^2 (\kappa + 1) - (\kappa - 1) \right] \left[\eta^2 - \xi^2 \right] \right. \\
& \quad + 3(\eta^2 - 1) - \frac{4\eta^2(\eta^2 - 1)}{\eta^2 - \xi^2} \left. \right] \left[(\eta^2 - \xi^2) \eta \ln \frac{\eta + 1}{\eta - 1} - 2\eta^2 \right] \\
& \quad + 2\xi^2 \left[\frac{\eta^2 - \xi^2}{\eta^2 - 1} + 2 \left(4\eta^2 - \xi^2 - 1 - \frac{4\eta^2(\eta^2 - 1)}{\eta^2 - \xi^2} \right) \right] \left. \right\} \quad (37)
\end{aligned}$$

(c) Solution of the Differential Equation

In order to solve (37), the equation is broken into two differential equations.

$$\begin{aligned}
& (\eta^2 - 1) \psi_{\eta\eta I}^{(2)} + (1 - \xi^2) \psi_{\xi\xi I}^{(2)} = \frac{\eta(1 - \xi^2)}{(\eta^2 - \xi^2)^2} \left[\xi^4 - 5\xi^2\eta^2 \right. \\
& \quad + 3\xi^2 + \eta^2 \left. \right] \left(\ln \frac{\eta + 1}{\eta - 1} - \frac{2\eta}{\eta^2 - \xi^2} \right) (\eta^2 - 1) \psi_{\eta\eta II}^{(2)} \\
& \quad + (1 - \xi^2) \psi_{\xi\xi II}^{(2)} = \frac{2\xi^2(1 - \xi^2)}{(\eta^2 - \xi^2)^2 3(\eta^2 - 1)} \left[(\eta^2 - \xi^2)^2 \right. \\
& \quad + 2(\eta^2 - 1) (\xi^4 - 5\xi^2\eta^2 + 3\eta^2 + \xi^2) \left. \right] \quad (38)
\end{aligned}$$

and determines one particular solution for each of the two. This is accomplished by solving the equation for individual terms on the right-hand side until finally all terms on the right are exhausted. In this way one obtains as particular integrals, with the simplification $(\ln x)^2 = \ln^2 x$

$$\begin{aligned}
 \psi_I^{(2)} = & - \frac{\eta^2(\eta^2 - 1)(1 - \xi^2)}{(\eta^2 - \xi^2)^2} \\
 & + \frac{1 - \xi^2}{\eta^2 - \xi^2} \left(-\frac{1}{2} + \eta^2 + \frac{1}{2} \eta (\eta^2 - 1) \ln \frac{\eta + 1}{\eta - 1} \right) \\
 & - (1 - \xi^2) \left[\frac{1}{8} (\eta^2 - 1) \ln^2 \frac{\eta + 1}{\eta - 1} + \frac{1}{2} \right] \\
 \psi_{II}^{(2)} = & \frac{2\eta^2(\eta^2 - 1)(1 - \xi^2)}{(\eta^2 - \xi^2)^2} \\
 & + \frac{1 - \xi^2}{\eta^2 - \xi^2} \left(1 - 4\eta^2 + \eta \ln \frac{\eta + 1}{\eta - 1} \right) + \frac{1}{2} \ln \frac{\eta^2 - 1}{\eta^2 - \xi^2} \\
 & + (1 - \xi^2) \left[-\frac{\eta}{2} \ln \frac{\eta + 1}{\eta - 1} + \frac{1}{8} (\eta^2 - 1) \ln^2 \frac{\eta + 1}{\eta - 1} + \frac{5}{2} \right]
 \end{aligned} \tag{39}$$

This solution substituted in (38) gives exactly the terms on the right-hand side. The indicated solution vanishes as $1/\eta^2$ aside from a constant at infinity, hence at infinity parallel flow is attained. To satisfy the boundary condition $\psi^{(2)} = 0$ at the contour $\eta = \eta_0$, solutions of the homogeneous differential equations must be determined which added to the nonhomogeneous solution makes the latter equal to zero, that is, cancels the previously obtained terms (39) for $\eta = \eta_0$. This is accomplished without difficulty for the terms of the form $(1 - \xi^2) \varphi(\eta)$ and

$\frac{1 - \xi^2}{\eta^2 - \xi^2} \dot{\varphi}(\eta)$, now we set up the equation

$$\psi = (1 - \xi^2) - \frac{\varphi(\eta)}{(\eta^2 - \xi^2)^n}$$

If the differential operator at the left-hand side of (38) is to disappear, the condition

$$\left[(\eta^2 - 1) \varphi'' - 2(n - 1)(2n - 1)\varphi \right] + 4n(\eta^2 - \xi^2) \left\{ -\eta(\eta^2 - 1) \varphi' + \left[(2n - 1) \eta^2 - n \right] \varphi \right\} = 0$$

will exist. For $n = 0$ the differential equation is

$$(\eta^2 - 1) \varphi'' - 2\varphi = 0$$

that is,

$$\varphi(\eta) = C_1 \left[\frac{1}{2} (\eta^2 - 1) \ln \frac{\eta + 1}{\eta - 1} - \eta \right] + C_2 (\eta^2 - 1)$$

If $n = 1$, both brackets must disappear, since the solution $\varphi(\eta)$ is to be independent of ξ . The equations $\varphi'' = 0$ and $\eta\varphi' - \varphi = 0$ are both satisfied by $\varphi = C_3\eta$. Hence one obtains as homogeneous basic solutions:

$$\psi_1 = (1 - \xi^2) (\eta^2 - 1), \psi_2 = (1 - \xi^2) \left[\frac{1}{2} (\eta^2 - 1) \ln \frac{\eta + 1}{\eta - 1} - \eta \right] \quad (40)$$

and

$$\psi_3 = (1 - \xi^2) \frac{\eta}{\eta^2 - \xi^2}$$

But for $n = 2$ two differential equations are obtained which have no common solution. Thus for the terms of the type

$$\frac{1 - \xi^2}{(\eta^2 - \xi^2)^2} \varphi(\eta)$$

and

$$\frac{1}{2} \ln \frac{\eta^2 - 1}{\eta^2 - \xi^2}$$

no closed solution complying with the homogeneous equation can be given, because the second expression, also which can be reduced to the first by differentiation, is of the same type. These expressions must therefore be developed in terms of functions which are solutions of the homogeneous equations. The general solution of the homogeneous equation which disappears for $\eta \rightarrow \infty$ is of the form

$$\psi = \sum_n \frac{c_n}{n(n+1)} (1 - \xi^2) \frac{dP_n(\xi)}{d\xi} (\eta^2 - 1) \frac{dQ_n(\eta)}{d\eta}$$

where $P_n(\xi)$ and $Q_n(\eta)$ represent the spherical functions of the first and second type. Thus the above expressions must be developed in series of this type for $\eta = \eta_0$. A homogeneous solution which for $\eta = \eta_0$ assumes the value $-\frac{1}{2} \ln \frac{\eta_0^2 - 1}{\eta_0^2 - \xi^2}$ is obtained (appendix, section b) by setting $\frac{dP_n(\xi)}{d\xi} = P_n'(\xi)$, $\frac{dQ_n(\eta)}{d\eta} = Q_n'(\eta)$

$$\psi_4 = \sum (1) = \sum_{n=0}^{\infty} \frac{4n+3}{(2n+1)(2n+2)} \frac{Q_{2n+1}'(\eta_0)}{(\eta_0^2 - 1) Q_{2n+1}(\eta_0)} (1 - \xi^2) P_{2n+1}'(\xi) (\eta^2 - 1) Q_{2n+1}'(\eta) \quad (41)$$

Since this series converges absolutely and uniformly for $\eta \geq \eta_0$ and $|\xi| \leq 1$ (see appendix, section b) for $\eta = \eta_0$

$$\psi_4(\eta_0) = -\frac{1}{2} \ln \frac{\eta_0^2 - 1}{\eta_0^2 - \xi^2}$$

Differentiating twice with respect to η gives a new series with the aid of which the second expression can be also represented (appendix, section b).

$$\sum (2) = \sum_{n=0}^{\infty} (4n+3) \frac{Q_{2n+1}(\eta_0)}{(\eta_0^2 - 1) Q_{2n+1}'(\eta_0)} (1 - \xi^2) P_{2n+1}'(\xi) (\eta^2 - 1) Q_{2n+1}'(\eta) \quad (42)$$

The series also converges uniformly and absolutely in the same range and two homogeneous solutions give

$$\psi_5(\eta_0) = \frac{\eta}{2} \frac{1 - \zeta^2}{\eta^2 - \zeta^2} \frac{\eta_0^2}{2} \sum^{(2)}$$

and hence for $\eta = \eta_0$

$$\psi_5(\eta_0) = \eta_0^3 \frac{1 - \zeta^2}{(\eta_0^2 - \zeta^2)^2}$$

Having thus the terms of the inhomogeneous solution supplemented by homogeneous terms with corresponding constants so that the expressions for $\eta = \eta_0$ disappear, these expressions are then appropriately expanded in powers of $\eta - \eta_0$. The first term then gives the derivatives of the two parts of the stream function with respect to η at $\eta = \eta_0$, needed for computing the maximum velocities at the contour alone:

$$\begin{aligned} \frac{\partial \psi_I(2)}{\partial \eta} \bigg|_{\eta=\eta_0} &= (1 - \zeta^2) \left[\frac{\frac{1}{4} \eta_0^2 \ln^2 \frac{\eta_0 + 1}{\eta_0 - 1} - 1}{\frac{1}{2} (\eta_0^2 - 1) \ln \frac{\eta_0 + 1}{\eta_0 - 1} - \eta_0} + \frac{1}{\eta_0^2 - \zeta^2} \frac{1 + 2\eta_0^3 \ln \frac{\eta_0 + 1}{\eta_0 - 1}}{2\eta_0} - \frac{F_1(\eta_0, \zeta)}{(\eta_0^2 - \zeta^2)^2} \right] \\ \frac{\partial \psi_{II}(2)}{\partial \eta} \bigg|_{\eta=\eta_0} &= (1 - \zeta^2) \left[\frac{\frac{1}{4} \ln^2 \frac{\eta_0 + 1}{\eta_0 - 1} - 2\eta_0 \ln \frac{\eta_0 + 1}{\eta_0 - 1} + 4 - \frac{1}{\eta_0^2 - 1}}{\frac{1}{2} (\eta_0^2 - 1) \ln \frac{\eta_0 + 1}{\eta_0 - 1} - \eta_0} - \frac{1}{\eta_0^2 - \zeta^2} \frac{3 + 4\eta_0^2 + \frac{2}{\eta_0^2 - 1}}{\eta_0} \right. \\ &\quad \left. + \frac{2F_1(\eta_0, \zeta)}{\eta_0^2 - \zeta^2} + \frac{\eta_0}{(\eta_0^2 - 1)(\eta_0^2 - \zeta^2)} + \frac{\partial}{\partial \eta} \sum_{\eta=\eta_0}^{(1)} \right] \quad (43) \end{aligned}$$

with

$$F_1(\eta_0, \xi) = 5\eta_0^3 - 3\eta_0 - \frac{1}{2\eta_0}(\eta_0^2 - 1)(\eta_0^2 - \xi^2) - 4\eta_0^3 \frac{\eta_0^2 - 1}{\eta_0^2 - \xi^2} \\ - \frac{1}{2}(\eta_0^2 - \xi^2)^2 \left[\frac{\partial}{\partial \eta} \sum_{\eta=\eta_0}^{(2)} \right] (\eta_0^2 - 1)$$

and

$$\frac{\partial}{\partial \eta} \sum^{(1)} = \sum (4n+3) \frac{Q_{2n+1}(\eta_0)}{(\eta_0^2 - 1) Q'_{2n+1}(\eta_0)} (1 - \xi^2) P'_{2n+1}(\xi) Q_{2n+1}(\eta)$$

$$\frac{\partial}{\partial \eta} \sum^{(2)} = \sum (4n+3)(2n+1)(2n+2) \\ \frac{Q_{2n+1}(\eta_0)}{(\eta_0^2 - 1) Q'_{2n+1}(\eta_0)} (1 - \xi^2) P'_{2n+1}(\xi) Q_{2n+1}(\eta)$$

The derivatives of the series follow from (41) and (42) when observing the relation $\frac{d}{d\eta} [(\eta^2 - 1) Q'_n(\eta)] = n(n+1) Q_n(\eta)$ (reference 4). From the derivation of the stream function $\psi^{(2)}$

$$\frac{\partial \psi^{(2)}}{\partial \eta} = - \frac{vU^*}{a_\infty^2} 2c^2 k^2 \left\{ \left[u^2(\kappa+1) - (\kappa-1) \frac{\partial \psi_I^{(2)}}{\partial \eta} = \frac{\partial \psi_{II}^{(2)}}{\partial \eta} \right] \right\} \\ c^2 = \frac{u^4}{v^2} U^*/4 \frac{(\eta_0^2 - 1)^2}{\left[\eta_0 - \frac{1}{2}(\eta_0^2 - 1) \ln \frac{\eta_0 + 1}{\eta_0 - 1} \right]^2}$$

there results the velocity according to (31) by means of the following relation for the ellipsoid center ($\xi = 0$)

$$\mu^2 \frac{1}{r} \frac{\partial \psi_I}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \tau} (\xi=0) \xrightarrow{\quad} \frac{1}{k^2 \eta} \frac{\partial \psi}{\partial \eta}$$

since from (6) follows

$$0 = k(\xi_T \eta + \eta_T \xi) \left(\frac{\partial}{\partial \xi} \right)_{\xi=0} k \eta \xi_T$$

$$1 = k \left(- \frac{\xi}{\sqrt{1 - \xi^2}} \xi_T \sqrt{\eta^2 - 1} + \sqrt{1 - \xi^2} \frac{\eta}{\sqrt{\eta^2 - 1}} \eta_T \right) \left(\frac{\partial}{\partial \xi} \right)_{\xi=0} k \frac{\eta}{\sqrt{\eta^2 - 1}} \eta_T$$

by differentiation, hence with $\tau \left(\frac{\partial}{\partial \xi} \right)_{\xi=0} k \sqrt{\eta^2 - 1}$

$$\frac{\psi_T}{\tau} \xrightarrow{\xi=0} \frac{1}{k \sqrt{\eta^2 - 1}} (\psi_\xi \xi_T + \psi_\eta \eta_T) \rightarrow \frac{1}{k^2} \frac{\psi_\eta}{\eta}$$

The correction $u_1^{(2)}$ for the velocity in the center of the ellipsoid therefore is in first approximation

$$\frac{u_{1\max}^{(2)}}{U^*} \approx - \frac{v \mu^2 \psi_{1r}^{(2)}}{U^* r} = \mu^4 \frac{M_\infty^2}{2} \left[\frac{\eta_0^2 - 1}{\eta_0 - (\eta_0^2 - 1) Q_0} \right]^2$$

$$\left\{ \left[\mu^2 (\kappa + 1) - (\kappa - 1) \right] \frac{\psi_{1\eta}^{(2)}}{\eta} \Big|_0 + \frac{\psi_{11\eta}^{(2)}}{\eta} \Big|_0 \right\} \quad (44)$$

Substituting in its stead the first approximation $\psi_1^{(1)}$ in (31) for the velocities, one obtains

$$\begin{aligned}
\frac{\psi_T(1)}{\tau} \xrightarrow[\eta=\eta_0]{\xi=0} \frac{1}{k^2} \frac{\psi_\eta(1)}{\eta} \Big/_0 &= \frac{c}{\eta_0} \left(\eta_0 \ln \frac{\eta_0 + 1}{\eta_0 - 1} - \frac{2\eta_0^2}{\eta_0^2 - 1} - 1 + \frac{\eta_0^2 + 1}{\eta_0^2 - 1} \right) \\
&= -\frac{\mu^2}{v} U^* \frac{\left(Q_0 - \frac{1}{\eta_0}\right)(\eta_0^2 - 1)}{\eta_0 - (\eta_0^2 - 1)Q_0} \frac{u_{1\max}(1)}{U^*} \\
&\approx -\frac{v}{U^*} \frac{\psi_T}{\tau} + v^2 M_\infty^2 \left(\frac{\psi_T}{\tau}\right)^2 \left[1 + \frac{\mu^2}{2} (1 + \kappa M_\infty^2) \right] \\
&= \mu^2 \frac{\left(Q_0 - \frac{1}{\eta_0}\right)(\eta_0^2 - 1)}{\eta_0 - (\eta_0^2 - 1)Q_0} + \mu^4 U^{*2} M_\infty^2 \left[1 + \frac{\mu^2}{2} (1 \right. \\
&\quad \left. + \kappa M_\infty^2) \right] \frac{(\eta_0^2 - 1)^2 \left(Q_0 - \frac{1}{\eta_0}\right)^2}{[\eta_0 - (\eta_0^2 - 1)Q_0]^2} \quad (45)
\end{aligned}$$

The total interference velocity in the center of the ellipsoid is thus

$$\begin{aligned}
\frac{u_{1\max}}{U^*} &\approx \mu^2 \left(Q_0 - \frac{1}{\eta_0}\right) \frac{\eta_0^2 - 1}{\eta_0 - (\eta_0^2 - 1)Q_0} + \frac{\mu^4 M_\infty^2}{2} \left[\frac{\eta_0^2 - 1}{\eta_0 - (\eta_0^2 - 1)Q_0} \right]^2 \left\{ \left[2 \right. \right. \\
&\quad \left. \left. + \mu^2 (1 + \kappa M_\infty^2) \right] \left(Q_0 - \frac{1}{\eta_0}\right)^2 \right. \\
&\quad \left. + \left[1 + \mu^2 (1 + \kappa M_\infty^2) \right] \frac{\psi_{I\eta}(2)}{\eta} \Big/_0 + \frac{\psi_{II\eta}(2)}{\eta} \Big/_0 \right\} \quad (46)
\end{aligned}$$

Substituting in it the derivative of the stream function $\psi_1^{(2)}$, according to (43), it is seen that in the practical evaluation of the formulas only the calculation of the two infinite series

offers any difficulties. The derivatives of the spherical functions $P_n(\xi)$ can be given at once for $\xi = 0$, when bearing in mind that

$$(\xi^2 - 1) P'_n(\xi) = n(\xi P_n - P_{n-1}) \quad (\text{reference 4})$$

one obtains

$$\begin{aligned} P'_{2n+1}(0) &= (2n+1) P_{2n}(0) = (2n+1) \frac{1 \times 3 \times 5 \dots (2n-1)}{2 \times 4 \times 6 \dots 2n} (-1)^n \\ &= (-1)^n \frac{n+1}{2^{2n}} \left(\frac{2n+1}{n} \right) \end{aligned} \quad (47)$$

The spherical functions Q_n for small n are best computed by the recursion formula

$$n Q_n = (2n-1) \eta_0 Q_n - 1 - (n-1) Q_{n-2}$$

with

$$Q_0 = \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1}$$

$$Q_1 = \eta_0 Q_0 - 1$$

(48)

while the derivative Q'_n is obtained from the difference of two Q_n :

$$(\eta_0^2 - 1) Q'_n = n(\eta_0 Q_n - Q_{n-1}) \quad (49)$$

The convergence of the series $\frac{\partial}{\partial \eta} \sum (1)$ and $\frac{\partial}{\partial \eta} \sum (2)$ grows worse as the ellipsoid becomes more slender and the Mach number greater, that is, the closer η_0 approaches unity. Since these values are of particular interest, a larger number of series terms must be calculated. However, even in the calculation of $Q_{12}^{(2)}$ to six

valid digits by the recursion formula the initial value $Q_0^{(2)}$ to 20 digits must be exactly known, while, to obtain $\frac{\partial}{\partial \eta} \sum^{(2)}$ for three digits exact, Q_n to Q_{19} is required. The higher Q_n with $n > 10$ must therefore be determined by a well converging series. As the conventional representation of the $Q_n(\eta_0)$ as power series in terms of $\frac{1}{\eta_0}$ converges very slowly a new representation as hyper-geometrical series is used, which converges particularly well for higher n . (See appendix, section c).

The infinite series $\frac{\partial}{\partial \eta} \sum^{(1)}$ and $\frac{\partial}{\partial \eta} \sum^{(2)}$ computed this way for a number of η_0 values were approximated by curves, which pass well through the calculated values for the individual η_0 and for $\eta_0 \rightarrow 1$ and $\eta_0 \rightarrow \infty$ have the limiting values of the considered series (appendix, section d, (79)). The accurately computed values are given in table I, the approximation curves in figure 7.

TABLE I
CALCULATED VALUES FOR THE INFINITE SERIES

η_0	$\frac{\partial}{\partial \eta} \sum^{(1)}$	$\frac{\partial}{\partial \eta} \sum^{(2)}$	$\frac{1}{Q_0} \frac{\partial}{\partial \eta} \sum^{(1)}$	$\frac{1}{Q_0} \frac{\partial}{\partial \eta} \sum^{(2)}$
1	- - - - -	- - - - -	-1	+4
1.25	-0.493 237	0.081 508	-.450 784	.074 192
1.5	-.219 561	-.125 578	-.272 842	-.156 052
1.732	-.126 722	-.120 631	-.192 447	-.183 197
2	-.076 117	-.093 211	-.138 570	-.169 688
3	-.020 096	-.033 401	-.057 984	-.096 375

A closer analysis of the behavior of the infinite series in the vicinity of $\eta_0 = 1$ yields as asymptotic representation for small η_0 (≈ 1) (appendix, section d, (76) and (78) and $\xi = 0$.

$$\frac{1}{Q_0} \frac{\partial}{\partial \eta} \sum^{(1)} \approx -\frac{1}{\eta_0^2} \xrightarrow{\eta_0=1} -1, \quad \frac{1}{Q_0} \frac{\partial}{\partial \eta} \sum^{(2)} \approx 2 \frac{3 - \eta_0^2}{\eta_0^4} \xrightarrow{\eta_0=1} 4 \quad (50)$$

These approximations for η_0 valid near unity are also shown in figure 7 and it is seen that even small departures from $\eta_0 = 1$ are accompanied by perceptible differences in the values of the functions. An exact calculation therefore requires the use of the interpolation curves of the exact values even for small η_0 . But if only a rough estimate is required η_0 can be set equal to unity in the additional term of the second approximation except in Q_0 , and the asymptotic representation of the derived sums

$$\frac{\partial \sqrt{(1)}}{\partial \eta} \approx -Q_0; \quad \frac{\partial \sqrt{(2)}}{\partial \eta} \approx 4Q_0 \quad (50')$$

may be applied. In this case the expression for the maximum velocity at the ellipsoid can be simplified. Inserting (50') in (43) gives at $\xi = 0$, for the derivatives of the contributions of the stream function the asymptotically valid limiting curves (fig. 1)

$$\psi_{I\eta}^{(2)}/_0 \approx -Q_0^2 + 2Q_0 - \frac{1}{2}; \quad \psi_{II\eta}^{(2)}/_0 \approx -Q_0^2 + 4Q_0 - 5 \quad (51)$$

For η_0 near unity the following developments are valid

$$\eta_0 \approx 1 + \frac{1}{2} \left(\frac{\epsilon^*}{\mu} \right)^2 + \frac{3}{8} \left(\frac{\epsilon^*}{\mu} \right)^4; \quad \frac{1}{\eta_0} \approx 1 - \frac{1}{2} \left(\frac{\epsilon^*}{\mu} \right)^2 + \dots$$

$$\eta_0^2 - 1 \approx \left(\frac{\epsilon^*}{\mu} \right)^2 \left[1 + \left(\frac{\epsilon^*}{\mu} \right)^2 + \dots \right]$$

$$Q_0 \approx \frac{1}{2} \ln \left(\frac{2\mu}{\epsilon^*} \right)^2 - \left(\frac{\epsilon^*}{2\mu} \right)^2 \approx \ln \frac{2\mu}{\epsilon^*} + \dots$$

Inserting these expressions in (46) one notices that only the term of the lowest order needs to be considered in the parenthesis because the entire bracket is already proportional to $\left(\frac{\epsilon^*}{\mu} \right)^4$.

Therefore η_0 may be set equal to 1 everywhere in the bracket

with exception of the terms $Q_0 \approx \ln \frac{2\mu}{\epsilon^*}$; one thus obtains up to

the terms of the order $> \left(\frac{\epsilon^*}{\mu}\right)^4$, for the increase of velocity in the center of the ellipsoid, with $\mu^2 M_\infty^2 = \mu^2 - 1$

$$\begin{aligned} \left(\frac{u_1}{U^*}\right)_{\max} &\approx -\mu^2 \left(\frac{\epsilon^*}{\mu}\right)^2 \frac{\left[1 + \left(\frac{\epsilon^*}{\mu}\right)^2\right] \left[1 - \frac{1}{4} \left(\frac{\epsilon^*}{\mu}\right)^2 - \ln \frac{2\mu}{\epsilon^*}\right]}{1 + \frac{1}{2} \left(\frac{\epsilon^*}{\mu}\right)^2 - \left(\frac{\epsilon^*}{\mu}\right) \ln \frac{2\mu}{\epsilon^*}} \\ &\quad + \frac{\mu^4 M_\infty^2}{2} \left(\frac{\epsilon^*}{\mu}\right)^4 \left\{ \left[2 + \mu^2(1 + \kappa M_\infty^2)\right] \left(1 - 2 \ln \frac{2\mu}{\epsilon^*} + \ln^2 \frac{2\mu}{\epsilon^*}\right) \right. \\ &\quad \left. + \left[1 + \mu^2(1 + \kappa M_\infty^2)\right] \frac{\psi_{I\eta}}{\eta}_o + \frac{\psi_{II\eta}}{\eta}_o \right\} \\ &\approx -\mu^2 \left(\frac{\epsilon^*}{\mu}\right)^2 \left[1 + \frac{3}{4} \left(\frac{\epsilon^*}{\mu}\right)^2 - \ln \frac{2\mu}{\epsilon^*} - \left(\frac{\epsilon^*}{\mu}\right)^2 \ln \frac{2\mu}{\epsilon^*}\right] \left[1 - \frac{1}{2} \left(\frac{\epsilon^*}{\mu}\right)^2 \right. \\ &\quad \left. + \left(\frac{\epsilon^*}{\mu}\right)^2 \ln \frac{2\mu}{\epsilon^*}\right] + \frac{\mu^4 M_\infty^2}{2} \left(\frac{\epsilon^*}{\mu}\right)^4 \left[\frac{\mu^2}{2} (1 + \kappa M_\infty^2) + 2 \ln \frac{2\mu}{\epsilon^*} - \frac{7}{2}\right] \\ &\approx \epsilon^{*2} \left(\ln \frac{2\mu}{\epsilon^*} - 1\right) + \frac{\epsilon^{*4}}{2\mu^2} \left(2 \ln^2 \frac{2\mu}{\epsilon^*} - \ln \frac{2\mu}{\epsilon^*} - \frac{1}{2}\right) \\ &\quad + \frac{\epsilon^{*4}}{2\mu^2} \left[2(\mu^2 - 1) \ln \frac{2\mu}{\epsilon^*} - \frac{7}{2} (\mu^2 - 1) + \frac{\mu^2(\mu^2 - 1)}{2} (1 + \kappa M_\infty^2)\right] \end{aligned}$$

hence

$$\begin{aligned} \left(\frac{u_1}{U^*}\right)_{\max} &\approx \epsilon^{*2} \left(\ln \frac{2\mu}{\epsilon^*} - 1\right) + \frac{\epsilon^{*4}}{2\mu^2} \left[2 \ln^2 \frac{2\mu}{\epsilon^*} + (2\mu^2 - 3) \ln \frac{2\mu}{\epsilon^*} \right. \\ &\quad \left. + \frac{7}{2} \mu^2 + 3 + \frac{\mu^2(\mu^2 - 1)}{2} (1 + \kappa M_\infty^2)\right] \end{aligned} \quad (52)$$

III. SUMMARY

(a) The effect of compressibility on the axially symmetrical flow around an ellipsoid was determined by computing the maximum velocities as function of the Mach number $M_\infty = \frac{U^*}{c_\infty}$ (c_∞ = velocity of sound at free-stream velocity U^*) in second approximation for ellipsoids of various thicknesses. The values obtained are compared with the values of the first approximation which are obtained by computing the incompressible velocity values for a slender ellipsoid of thickness ratio $\frac{d}{t} = \sqrt{1 - M_\infty^2} \frac{d^*}{t}$ (d^* = thickness of ellipsoid in compressible flow) by the Prandtl-Göthert rule (reference 1) and the result is multiplied by the square of the factor

$$\mu = \frac{1}{\sqrt{1 - M_\infty^2}} \text{ opposed to the initial distortion.}$$

For small thicknesses $\left(\frac{d^*}{t} \leq 0.1\right)$ the second approximation almost coincides with the Prandtl approximation, and the differences are still small up to $\frac{d^*}{t} \leq 0.3$ (error ≤ 5 percent of disturbance velocity at $M_\infty = 0.8$ (fig. 2).

(b) A comparison with the maximum velocities for the sphere, calculated by Lamla with the fourth approximation of the Rayleigh method indicates that even in this extreme case the departures even of the second approximation are not appreciable (error ≈ 2.5 percent of the disturbance velocity at $M_\infty = 0.5$). Thus the second approximation is still amply satisfactory for thick ellipsoids $\left(\frac{d^*}{t} \leq 0.5\right)$ (fig. 3).

(c) The ratios of the interference velocities $u_{\text{compress.}} / u_{\text{incompress.}}$ and of the total velocities $u_{\text{compress.}} / u_{\text{incompress.}}$ are represented in diagrams (figs. 4 and 6) for the point of maximum thickness. The velocities of the second approximation calculated here can be obtained directly by these diagrams, if the incompressible values are taken from figure 8.

(d) Developing the first and second approximation for the maximum velocities in powers of the thickness ratio (the logarithmic quantity $\ln\left(\frac{d^*}{t}\right)$ being regarded as of the order of magnitude of unity) gives expressions of first and second approximations

asymptotically valid for small thickness, depending upon whether only terms proportional to $\left(\frac{d^*}{t}\right)^2$ or also those proportional to $\left(\frac{d^*}{t}\right)^4$ are included. For small thicknesses $\left(\frac{d^*}{t} \leq 0.2\right)$ the second approximation can be replaced by the clearer asymptotic expression without introducing an appreciable error. In this manner simpler formulas for the velocity ratios of compressible to incompressible flow are obtained. In first approximation one obtains for the location of maximum thickness

$$\frac{u_{\text{compress.}}}{u_{\text{incompress.}}} = 1 + \frac{\ln \mu}{\ln \frac{2}{\frac{d^*}{t}} - 1}$$

and

$$\frac{u_{\text{compress.}}}{u_{\text{incompress.}}} = 1 + \left(\frac{d^*}{t}\right)^2 \ln \mu$$

IV. APPENDIX

Auxiliary Calculations

(a) Transformation to elliptic coordinates. - The transformation (6)

$$\sigma = \alpha\gamma \quad \tau = \beta\delta$$

with

$$\begin{aligned} \alpha &= k \cosh \xi & \beta &= k \sinh \xi & \gamma &= \cos \theta & \delta &= \sin \theta \\ \alpha^2 - \beta^2 &= k^2 & \gamma^2 + \delta^2 &= 1 \end{aligned}$$

gives because of the Cauchy-Riemann differential equations

$$\begin{aligned}\sigma_{\xi}^{\eta} = \tau_{\theta} = \beta\gamma, \quad \sigma_{\theta} = -\tau_{\xi} = -\alpha\delta, \quad \sigma_{\xi\xi} - \tau_{\theta\xi} = -\sigma_{\theta\theta} = \alpha\gamma, \tau_{\xi\xi} = -\sigma_{\theta\xi} \\ = -\tau_{\theta\theta} = \beta\delta\end{aligned}$$

Conversely with

$$D = \beta^2\gamma^2 + \alpha^2\delta^2 = \beta^2 + k^2\delta^2 = \alpha^2 - k^2\gamma^2$$

$$\xi_{\sigma} = \theta_{\tau} = \frac{\beta\gamma}{D} \quad \theta_{\sigma} = -\xi_{\tau} = -\frac{\alpha\delta}{D}$$

$$\begin{aligned}\xi_{\sigma\sigma} = \xi_{\sigma\xi}\xi_{\sigma} + \xi_{\sigma\theta}\theta_{\sigma} = \frac{\alpha\gamma}{D^2} (D - 2\beta^2) \frac{\beta\gamma}{D} + \frac{\beta\delta}{D^2} (D - 2\alpha^2) \frac{-\alpha\delta}{D} \\ = \frac{\alpha\beta}{D^2} \left(\gamma^2 - \delta^2 + 2 \frac{\alpha^2\delta^2 - \beta^2\gamma^2}{D} \right) = \theta_{\tau\sigma} = -\xi_{\tau\tau}\end{aligned}$$

$$\begin{aligned}\xi_{\sigma\tau} = \xi_{\sigma\xi}\xi_{\tau} + \xi_{\sigma\theta}\theta_{\tau} = \frac{\alpha\gamma}{D^2} (D - 2\beta^2) \frac{\alpha\delta}{D} + \frac{\beta\delta}{D^2} (D - 2\alpha^2) \frac{\beta\gamma}{D} \\ = \frac{\gamma\delta}{D^2} \left(\alpha^2 + \beta^2 - \frac{4\alpha^2\beta^2}{D} \right) = \theta_{\tau\tau} = -\theta_{\sigma\sigma}\end{aligned}$$

For the stream function, then

$$\psi_{\sigma} = \psi_{\xi}\xi_{\sigma} + \psi_{\theta}\theta_{\sigma} = \frac{1}{D} (\beta\gamma\psi_{\xi} - \alpha\delta\psi_{\theta}); \quad \psi_{\tau} = \psi_{\xi}\xi_{\tau} + \psi_{\theta}\theta_{\tau} = \frac{1}{D} (\alpha\delta\psi_{\xi} + \beta\gamma\psi_{\theta})$$

and

$$\begin{aligned}
 \psi_{\sigma\sigma} &= (\psi_{\xi\xi}\xi_{\sigma} + \psi_{\xi\theta}\theta_{\sigma})\xi_{\sigma} + \psi_{\xi\xi}\xi_{\sigma\sigma} + (\psi_{\theta\xi}\xi_{\sigma} + \psi_{\theta\theta}\theta_{\sigma})\theta_{\sigma} + \psi_{\theta\theta}\theta_{\sigma\sigma} \\
 &= \frac{1}{D^2} \left[\psi_{\xi\xi} \beta^2 \gamma^2 + \psi_{\theta\theta} \alpha^2 \delta^2 - 2\psi_{\xi\theta} \alpha\beta\gamma\delta + \psi_{\xi\xi} \alpha\beta \left(\gamma^2 - \delta^2 + 2 \frac{\alpha^2 \delta^2 - \beta^2 \gamma^2}{D} \right) \right. \\
 &\quad \left. - \psi_{\theta\theta} \gamma\delta \left(\alpha^2 + \beta^2 - \frac{4\alpha^2 \beta^2}{D} \right) \right] \\
 \psi_{\sigma\tau} &= (\psi_{\xi\xi}\xi_{\tau} + \psi_{\xi\theta}\theta_{\tau})\xi_{\sigma} + \psi_{\xi\xi}\xi_{\sigma\tau} + (\psi_{\theta\xi}\xi_{\tau} + \psi_{\theta\theta}\theta_{\tau})\theta_{\sigma} + \psi_{\theta\theta}\theta_{\sigma\tau} \\
 &= \frac{1}{D^2} \left[\alpha\beta\gamma\delta (\psi_{\xi\xi} - \psi_{\theta\theta}) + \psi_{\xi\theta} (\beta^2 \gamma^2 - \alpha^2 \delta^2) + \psi_{\xi\xi} \gamma\delta \left(\alpha^2 + \beta^2 - \frac{4\alpha^2 \beta^2}{D} \right) \right. \\
 &\quad \left. + \psi_{\theta\theta} \alpha\beta \left(\gamma^2 - \delta^2 + 2 \frac{\alpha^2 \delta^2 - \beta^2 \gamma^2}{D} \right) \right]
 \end{aligned} \tag{53}$$

The left side of the differential equation (34) thus becomes since $\psi_{\sigma\sigma} + \psi_{\tau\tau} = \frac{1}{D} (\psi_{\xi\xi} + \psi_{\theta\theta})$

$$\Delta^* \psi^{(2)} = \frac{1}{D} \psi_{\xi\xi}^{(2)} + \psi_{\theta\theta}^{(2)} - \frac{\alpha}{\beta} \psi_{\xi\xi}^{(2)} - \frac{\gamma}{\delta} \psi_{\theta\theta}^{(2)} \tag{54}$$

while

$$\psi^{(1)} = K\beta^2\delta^2 \left(\frac{1}{2} \ln \frac{\alpha+k}{\alpha-k} - \frac{\alpha k}{\beta^2} \right) K = \frac{-U/2}{\frac{\eta_0}{\eta_0^2 - 1} - \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1}}$$

must be inserted on the right-hand side of (34), so that

$$\alpha\beta\psi_{\xi}^{(1)} = K\beta^2\delta^2 \left(\alpha^2 \ln \frac{\alpha+k}{\alpha-k} - 2\alpha k \right) \quad \gamma\delta\psi_{\theta}^{(1)} = K\gamma^2\delta^2 \left(\beta^2 \ln \frac{\alpha+k}{\alpha-k} - 2\alpha k \right)$$

$$\psi_{\xi\xi}^{(1)} = K\delta^2 \left[(\alpha^2 + \beta^2) \ln \frac{\alpha+k}{\alpha-k} - 4\alpha k \right]$$

$$\psi_{\theta\theta}^{(1)} = K(\gamma^2 - \delta^2) \left(\beta^2 \ln \frac{\alpha+k}{\alpha-k} - 2\alpha k \right)$$

$$\psi_{\xi\theta}^{(1)} = K\gamma\delta \left(2\alpha\beta \ln \frac{\alpha+k}{\alpha-k} - 4\beta k \right)$$

Inserted in (53) one obtains

$$\begin{aligned} \psi_{\sigma\sigma}^{(1)} &= \frac{K}{D^2} \left\{ \beta^2\delta^2 \ln \frac{\alpha+k}{\alpha-k} \left[(\alpha^2 + \beta^2)\gamma^2 + (\gamma^2 - \delta^2)\alpha^2 - 4\alpha^2\gamma^2 \right. \right. \\ &\quad \left. \left. + \alpha^2(\gamma^2 - \delta^2) + 2\alpha^2 \frac{\alpha^2\delta^2 - \beta^2\gamma^2}{D} - \gamma^2(\alpha^2 + \beta^2) + 4 \frac{\alpha^2\beta^2\gamma^2}{D} \right] \right. \\ &\quad \left. - 2k\alpha\delta^2 \left[2\beta^2\gamma^2 + (\gamma^2 - \delta^2)\alpha^2 - 4\beta^2\gamma^2 + \beta^2(\gamma^2 - \delta^2) \right. \right. \\ &\quad \left. \left. + 2\beta^2 \frac{\alpha^2\delta^2 - \beta^2\gamma^2}{D} - \gamma^2(\alpha^2 + \beta^2) + 4 \frac{\alpha^2\beta^2\gamma^2}{D} \right] \right\} \\ &= \frac{K}{D^2} 2k\alpha\delta^2 \left(D + 3\beta^2 - \frac{4\alpha^2\beta^2}{D} \right) \end{aligned}$$

$$\begin{aligned}
\psi_{\sigma\tau}(1) &= \frac{K}{D^2} \left\{ \alpha\beta\gamma\delta \ln \frac{\alpha+k}{\alpha-k} \left[\delta^2(\alpha^2 + \beta^2) - \beta^2(\gamma^2 - \delta^2) + 2(\beta^2\gamma^2 - \alpha^2\delta^2) \right. \right. \\
&\quad \left. \left. + \delta^2(\alpha^2 + \beta^2) - \frac{4\alpha^2\beta^2\delta^2}{D} + \beta^2(\gamma^2 - \delta^2) + 2\beta^2 \frac{\alpha^2\delta^2 - \beta^2\gamma^2}{D} \right] \right. \\
&\quad \left. - 2k\beta\gamma\delta \left[2\alpha^2\delta^2 - \alpha^2(\gamma^2 - \delta^2) + 2(\beta^2\gamma^2 - \alpha^2\delta^2) + \delta^2(\alpha^2 + \beta^2) \right. \right. \\
&\quad \left. \left. - \frac{4\alpha^2\beta^2\delta^2}{D} + \alpha^2(\gamma^2 - \delta^2) + 2\alpha^2 \frac{\alpha^2\delta^2 - \beta^2\gamma^2}{D} \right] \right\} \\
&= -\frac{K}{D^2} 2k\beta\gamma\delta \left(D + 2\alpha^2 + \beta^2 - \frac{4\alpha^2\beta^2}{D} \right)
\end{aligned}$$

$$\frac{\psi_{\tau}(1)}{\tau} = \frac{K}{D} \left(D \ln \frac{\alpha+k}{\alpha-k} - 2\alpha k \right); \quad \frac{\psi_{\sigma}(1)}{\tau} = 2 \frac{K}{D} \frac{k^3\gamma\delta}{\beta}$$

The thus obtained expressions can be immediately transformed into elliptic coordinates. There is:

$$\frac{D}{k^2} = \eta^2 - \xi^2; \quad \text{since } \frac{\alpha}{k} = \eta; \quad \gamma = \xi, \quad \text{and } \frac{\beta^2}{k^2} = \eta^2 - 1; \quad \delta^2 = 1 - \xi^2$$

Now, since

$$\psi_{\xi} = \psi_{\eta} \frac{\beta}{k}; \quad \psi_{\theta} = -\psi_{\xi}\delta; \quad \psi_{\xi\xi} = \psi_{\eta\eta} \frac{\beta^2}{k^2} + \psi_{\eta} \frac{\alpha}{k}$$

$$\psi_{\theta\theta} = \psi_{\xi\xi} \delta^2 - \psi_{\xi}\gamma$$

The left side of the differential equation becomes

$$\Delta^*\psi^{(2)} = \frac{1}{k^2(\eta^2 - \xi^2)} \left[(\eta^2 - 1)\psi_{\eta\eta} + (1 - \xi^2)\psi_{\xi\xi} \right]$$

and the expressions on the right-hand side are

$$D \frac{\psi_{1\tau}}{\tau} \psi_{1\sigma\sigma} = \frac{K^2 k^2}{(\eta^2 - \xi^2)^2} 2\eta(1 - \xi^2) \left[\eta^2 - \xi^2 + 3(\eta^2 - 1) \right. \\ \left. - \frac{4\eta^2(\eta^2 - 1)}{\eta^2 - \xi^2} \right] \left[(\eta^2 - \xi^2) \ln \frac{\eta + 1}{\eta - 1} - 2\eta \right] \\ D \frac{\psi_{1\sigma}}{\tau} \left(\frac{\psi_{1\sigma}}{\tau} - 2\psi_{1\sigma\tau} \right) = \frac{K^2 k^2}{(\eta^2 - \xi^2)^2} 4\xi^2(1 - \xi^2) \left\{ \frac{\eta^2 - \xi^2}{\eta^2 - 1} \right. \\ \left. + 2 \left[\eta^2 - \xi^2 + 2\eta^2 + \eta^2 - 1 - \frac{4\eta^2(\eta^2 - 1)}{\eta^2 - \xi^2} \right] \right\}$$

Thus the differential equation reads:

$$(\eta^2 - 1)\psi_{\eta\eta}^{(2)} + (1 - \xi^2)\psi_{\xi\xi}^{(2)} = - \frac{vU^*}{\alpha_1^2} 2 \frac{K^2 k^2}{(\eta^2 - \xi^2)} (1 - \xi^2) \\ \left(\left[\mu^2(\kappa + 1) - (\kappa - 1) \right] \left[\eta^2 - \xi^2 + 3(\eta^2 - 1) \right. \right. \\ \left. \left. - \frac{4\eta^2(\eta^2 - 1)}{\eta^2 - \xi^2} \right] \left[(\eta^2 - \xi^2) \eta \ln \frac{\eta + 1}{\eta - 1} - 2\eta^2 \right] \right. \\ \left. + 2\xi^2 \left\{ \frac{\eta^2 - \xi^2}{\eta^2 - 1} + 2 \left[4\eta^2 - \xi^2 - 1 - \frac{4\eta^2(\eta^2 - 1)}{\eta^2 - \xi^2} \right] \right\} \right) \quad (55)$$

(b) Developments in spherical functions and convergence studies for the series. - To obtain the development of

$-\ln \frac{\eta^2 - 1}{\eta^2 - \xi^2}$ at $\eta = \eta_0$ in a series of the form

$$\psi_4 = \sum_{n=0}^{\infty} A_n(1 - \xi^2) P_n'(\xi)(\eta^2 - 1) Q_n'(\xi)$$

multiply

$$-\ln \frac{\eta_0^2 - 1}{\eta_0^2 - \xi^2} = \sum B_n (1 - \xi^2) P_n'(\xi)$$

with $P_n'(\xi)$ (where $B_n = A_n(\eta_0^2 - 1)Q_n'(\eta_0)$) one then obtains, owing to

$$\int_{-1}^1 (1 - \xi^2) P_n' P_m' d\xi = \begin{cases} 0 & n \neq m \\ \frac{2n(n+1)}{2n+1} & n = m \end{cases} \quad (\text{reference 5})$$

for the series coefficients by partial integration

$$\begin{aligned} B_n &= -\frac{2n+1}{2n(n+1)} \int_{-1}^1 P_n'(\xi) \ln \frac{\eta_0^2 - 1}{\eta_0^2 - \xi^2} d\xi \\ &= +\frac{2n+1}{n(n+1)} \int_{-1}^1 \frac{\xi P_n(\xi)}{\eta_0^2 - \xi^2} d\xi = \begin{cases} 0 & n = 2n' \\ 2Q_n(\eta_0) \frac{2n+1}{n(n+1)} & n = 2n' + 1 \end{cases} \end{aligned}$$

As for

$$\begin{aligned} Q_n(z) &= \frac{1}{2} \int_{-1}^1 P_n(\xi) \frac{d\xi}{z - \xi} = \frac{1}{2} \int_{-1}^1 P_n(-\xi) \frac{d\xi}{z + \xi} \\ &= \frac{(-1)^{n+1}}{2} \int_{-1}^1 P_n(\xi) \frac{d\xi}{z + \xi} \quad (\text{reference 5}) \end{aligned}$$

$$\int_{-1}^1 P_n(\xi) \frac{\xi d\xi}{z^2 - \xi^2} = \frac{1}{2} \int_{-1}^1 P_n(\xi) \left(\frac{1}{z - \xi} - \frac{1}{z + \xi} \right) d\xi = \begin{cases} 0 & n \text{ even} \\ 2Q_n(z) & n \text{ odd} \end{cases} \quad (56)$$

Hence the required homogeneous solution which has the desired value for $\eta = \eta_0$ is

$$\psi_4 = \sum_{n=0}^{\infty} (1) = \sum_{n=0}^{\infty} \frac{4n' + 3}{(2n' + 1)(2n' + 2)} \frac{Q_{2n'+1}(\eta_0)}{(\eta_0^2 - 1)Q'_{2n'+1}(\eta_0)}$$

$$(1 - \xi^2)P'_{2n'+1}(\xi)(\eta^2 - 1)Q_{2n'+1}(\eta) \xrightarrow{\eta=\eta_0} -\ln \frac{\eta_0^2 - 1}{\eta_0^2 - \xi^2} \quad (57)$$

To obtain the further development of $(1 - \xi^2) \frac{\eta^3}{(\eta_0^2 - \xi^2)^2}$, the homogeneous solution $\eta \frac{1 - \xi^2}{\eta^2 - \xi^2}$ is represented by a series that must converge for all $\eta > 1$:

$$\frac{\eta(1 - \xi^2)}{\eta^2 - \xi^2} = \sum A_n^* (1 - \xi^2) P_n'(\xi) (\eta^2 - 1) Q_n'(\eta)$$

For $\eta = \eta_0$ follows

$$\frac{\eta_0(1 - \xi^2)}{\eta_0^2 - \xi^2} = \sum B_n^* (1 - \xi^2) P_n'(\xi)$$

with

$$B_n^* = A_n^* (\eta_0^2 - 1) Q_n'(\eta_0)$$

and as before, the coefficients follow by partial integration as

$$B_n^* = \frac{2n+1}{2n(n+1)} \eta_0 \int_{-1}^1 P_n'(\xi) \frac{1 - \xi^2}{\eta_0^2 - \xi^2} d\xi$$

$$= \frac{2n+1}{2n(n+1)} \eta_0 \int_{-1}^1 \frac{2\xi(\eta_0^2 - 1)}{(\eta_0^2 - \xi^2)^2} P_n(\xi) d\xi$$

$$= \begin{cases} 0 & n = 2n' \\ -\frac{2n+1}{n(n+1)} (\eta_0^2 - 1) Q_n'(\eta_0) & n = 2n' + 1 \end{cases}$$

By (56), integration with respect to z gives

$$\int_{-1}^1 P_n(\xi) \frac{\xi z d\xi}{(z^2 - \xi^2)^2} = \begin{cases} 0 & n \text{ even} \\ -Q_n'(z) & n \text{ odd} \end{cases} \quad (58)$$

Hence for all $\eta > 1$ the series development

$$\frac{\eta(1 - \xi^2)}{\eta^2 - \xi^2} = - \sum_{n'=0}^{\infty} \frac{4n' + 3}{(2n' + 1)(2n' + 2)} (1 - \xi^2) P'_{2n'+1}(\xi) (\eta^2 - 1) Q'_{2n'+1}(\eta) \quad (59)$$

is valid. Since

$$\frac{d}{d\eta} [(\eta^2 - 1) Q'_{2n'+1}] = 2(n' + 1)(2n' + 1) Q_{2n'+1}$$

it follows by differentiation which is permissible for $\eta \geq \eta_0$ that

$$\frac{1 - \xi^2}{\eta^2 - \xi^2} - \frac{2\eta^2(1 - \xi^2)}{(\eta^2 - \xi^2)^2} = - \sum_{n'=0}^{\infty} (4n' + 3)(1 - \xi^2) P'_{2n'+1}(\xi) Q_{2n'+1}(\eta) \quad (60)$$

For $\eta = \eta_0$ there follows multiplication with η_0 from (60) and subtraction of (59),

$$\frac{\eta_0^3(1 - \xi^2)}{(\eta_0^2 - \xi^2)^2} = \sum_{n'=0}^{\infty} \frac{1}{\eta_0^2} \frac{4n' + 3}{(2n' + 1)(2n' + 2)} (1 - \xi^2) P'_{2n'+1}(\xi) \left[\eta_0 Q_{2n'+1}(\eta_0)(2n' + 1)(2n' + 2) - (\eta_0^2 - 1) Q'_{2n'+1}(\eta_0) \right]$$

On the other hand the following development is to be valid

$$\frac{\eta_o^3(1 - \xi^2)}{(\eta_o^2 - \xi^2)^2} = \sum_0^{\infty} A_{2n'+1}^{**} (1 - \xi^2) P'_{2n'+1}(\xi) (\eta_o^2 - 1) Q'_{2n'+1}(\eta_o) = \sum_0^{\infty} B_n^{**} (1 - \xi^2) P_n'(\xi) \quad (61)$$

By comparison of the coefficients one therefore obtains owing to the Legendre differential equation (reference 5)

$$\left. \begin{aligned} \eta_o Q'_{2n'+1}(\eta_o) (2n' + 1)(2n' + 2) &= 2\eta_o^2 Q'_{2n'+1}(\eta_o) + \eta_o(\eta_o^2 - 1) Q''_{2n'+1}(\eta_o) \\ A_{2n'+1}^{**} &= \frac{1}{2} \frac{4n' + 3}{(2n' + 1)(2n' + 2)} \frac{(\eta_o^2 + 1) Q'_{2n'+1}(\eta_o) + \eta_o(\eta_o^2 - 1) Q''_{2n'+1}(\eta_o)}{(\eta_o^2 - 1) Q'_{2n'+1}(\eta_o)} \end{aligned} \right\} (62)$$

This development can also be secured directly by starting from (61) multiplying by $P_n'(\xi)$ and integrating over ξ . The integral on the left-hand side

$$\int_{-1}^1 \frac{\eta_o^3(1 - \xi^2)}{(\eta_o^2 - \xi^2)^2} P_n'(\xi) d\xi$$

gives after partial integration

$$\int_{-1}^1 \frac{2\xi\eta_o^3}{(\eta_o^2 - \xi^2)^2} P_n d\xi + \int_{-1}^1 \frac{4\eta_o^3(\eta_o^2 - 1)}{(\eta_o^2 - \xi^2)^3} \xi P_n d\xi$$

and these integrals can be expressed according to (58) by the derivative of $Q_n(z)$. By differentiation for $n = 2n' + 1$, (58) gives

$$Q_n''(z) = - \int_{-1}^1 \xi P_n(\xi) \left[\frac{1}{(z^2 - \xi^2)^2} - \frac{4z^2}{(z^2 - \xi^2)^3} \right] d\xi \quad (63)$$

hence

$$P_n'' \frac{2n(n+1)}{2n+1} = 2\eta_0^2 Q_n' - (\eta_0^2 - 1)Q_n' + \eta_0(\eta_0^2 - 1)Q_n''$$

thus the relation (62) is obtained. Adding the series (59) and dividing the series (61) by -2 for arbitrary η one finally obtains

$$\psi_5 = \frac{n}{2} \frac{1 - \xi^2}{\eta^2 - \xi^2} + \sum_{0}^{\infty} \frac{4n+3}{2} \frac{\eta_0 Q_{2n+1}(\eta_0)}{(\eta_0^2 - 1)Q'_{2n+1}(\eta_0)}$$

$$(1 - \xi^2)P'_{2n+1}(\xi) (\eta^2 - 1)Q'_{2n+1}(\eta) \xrightarrow{\eta=\eta_0} \frac{\eta_0^3(1 - \xi^2)}{(\eta_0^2 - \xi^2)^2} \quad (64)$$

Whether the obtained series $\sum (1)$ and $\sum (2)$ actually attain the values given in (57) and (64) must be proved by absolute and uniform convergence. Differentiation by terms gives the $\frac{\partial}{\partial \eta} \sum (1)$ and $\frac{\partial}{\partial \eta} \sum (2)$ needed for the velocities in (43), the convergence of which must also be proved. It is seen that the (1) series differ from the (2) series only by the fact that

$$\alpha_n^{(1)} = \frac{\alpha_n^{(2)}}{(2n+1)(2n+2)} < \alpha_n^{(2)}$$

is valid for the n th term α_n . Therefore only the convergence for the (2) series needs to be proved. To this end, the spherical

functions are to be estimated first. From the integral representation for $Q_n(z)$ (reference 5) there follows:

$$\begin{aligned} Q_n(z) &= \frac{1}{2^{n+1}} \int_{-1}^1 \frac{(1-t^2)^n}{(z-t)^{n+1}} dt \\ &= \frac{1}{2^{n+1}} \int_{-1}^1 \frac{(1-t)^n}{(z-t)^n} \frac{z-t}{(z-t)^2} (1+t)^n dt \\ &< \frac{1}{2^{n+1}} \int_{-1}^1 \left(\frac{1-t}{z-t} \right)^n \frac{z+1}{(z-t)^2} 2^n dt \end{aligned}$$

Putting

$$\frac{1-t}{z-t} = u \quad \frac{du}{dt} = -\frac{z-1}{(z-t)^2}$$

and

$$\left(\frac{1-t}{z-t} \right)^n \frac{z+1}{(z-t)^2} dt = -\frac{z+1}{z-1} u^n du$$

gives

$$Q_n(z) < \frac{1}{2} \frac{z+1}{z-1} \frac{1}{n+1} \left(\frac{1-t}{z-t} \right)^{n+1} \Big|_{-1}^{-1} = \left(\frac{2}{z+1} \right)^n \frac{1}{(n+1)(z-1)} \quad (65)$$

and in addition

$$(z^2 - 1)Q_n'(z) = \frac{n+1}{2^{n+1}} \int_{-1}^1 \frac{(1-t^2)^n}{(z-t)^n} dt \quad (66)$$

For, from the differential equation of the spherical functions there follows by integration

$$\begin{aligned}
 (\eta_0^2 - 1)Q_n'(\eta_0) \frac{1}{\eta_0 - 1} - 1 (\eta_0^2 - 1)Q_n'(\eta_0) + 1 &= n(n+1) \int_{-1}^{\eta_0} Q_n(z) dz \\
 &= \frac{n(n+1)}{2^{n+1}} \int_1^{\eta_0} \int_{-1}^1 \frac{(1-t^2)^n}{(z-t)^{n+1}} dt dz \\
 &= \frac{n+1}{2^{n+1}} \int_{-1}^1 (1-t^2)^n \left[\frac{1}{(1-t)^n} - \frac{1}{(\eta_0-t)^n} \right] dt \\
 &= 1 - \frac{n+1}{2^{n+1}} \int_{-1}^1 \left(\frac{1-t^2}{\eta_0-t} \right)^n dt
 \end{aligned}$$

hence

$$\begin{aligned}
 |(z^2 - 1)Q_n'(z)| &= \frac{n+1}{2^{n+1}} \int_{-1}^1 \frac{(1-t^2)^n}{(z-t)^{n+1}} (z-t) dt \\
 &> (n+1)(z-1)Q_n(z)
 \end{aligned} \tag{67}$$

and finally, since $|P_n(\xi)| \leq \frac{1}{2}$ at $\xi \leq 1$

$$(1-\xi^2) |P_n'(\xi)| = n(P_{n-1} - \xi P_n) < n \left(|P_{n-1}| + \xi |P_n| \right) < 2n \tag{68}$$

Observing that $z = \eta$ in the denominator in the integral (66), hence that

it is seen to be sufficient to prove the convergence of $\sum (2)$ for $\eta = \eta_0$. The same holds true for the series $\frac{\partial}{\partial \eta} \sum (2)$, as from the integral representation of Q_n there follows correspondingly

$$\frac{Q_{2n+1}(\eta)}{Q_{2n+1}(\eta_0)} < 1 \text{ for } \eta > \eta_0$$

Therefore, only the series

$$\left| \sum (2) \right|_{\eta=\eta_0} = \sum (4n+3) Q_{2n+1}(\eta_0) (1-\xi^2) P'_{2n+1}(\xi) = \sum \alpha_n$$

and

$$\left[\frac{\partial}{\partial \eta} \sum (2) \right]_{\eta=\eta_0} = \sum (2n+1)(2n+2)(4n+3)$$

$$\frac{Q_{2n+1}^2(\eta_0)}{(\eta_0^2 - \xi^2) Q'_{2n+1}(\eta_0)} (1-\xi^2) P'_{2n+1}(\xi) = \sum A_n$$

needs to be analyzed. From (65) and (68) there follows with

$$\frac{2}{\eta_0 + 1} = q < 1 \text{ for } \eta_0 > 1$$

$$|\alpha_n| < \frac{4n+3}{\eta_0 - 1} q^{2n+1} \frac{1}{(2n+2)} 2(2n+1) < \frac{4}{\eta_0 - 1} (2n+2) q^{2n+1}$$

Hence $\sum (2)$ is thus uniformly and absolutely convergent for $|\xi| \leq 1$ and $\eta \geq \eta_0$, because a converging comparative series with greater members can be given. Correspondingly owing to (67) and (68)

$$|A_n| < \frac{(2n+1)(2n+2)(4n+3)}{(2n+2)(\eta_0-1)} Q_{2n+1}(\eta_0)(1-\xi^2)P'_{2n+1}(\xi) \\ < (2n+1)(4n+3) \frac{q^{2n+1}}{(2n+2)(\eta_0-1)^2} 2(2n+1)$$

Hence

$$|A_n| < \frac{4}{(\eta_0-1)^2} (2n+3)(2n+2) q^{2n+1}$$

The comparative series however converges, with certainty, as second derivative of a power series for $q < 1$; thus the absolute and uniform convergence of $\frac{\partial}{\partial \eta} \sum (2)$ itself is proved.

(c) Calculation of the spherical functions $Q_n(z)$ for the arguments $|z| > 1$ and $n > 10$ by means of a hypergeometric series. - For $z > 1$ the integral representation

$$Q_n(z) = \int_0^\infty (z + \sqrt{z^2 - 1} \cosh \xi)^{-n-1} d\xi \quad (69)$$

is applicable. Introduction of a new variable X by

$$(z + \sqrt{z^2 - 1} \cosh \xi) (z - \sqrt{z^2 - 1} \cosh X) = 1$$

so that

$$\frac{d\xi}{dX} = z + \sqrt{z^2 - 1} \cosh \xi$$

gives

$$Q_n(z) = \int_0^{X_0} (z - \sqrt{z^2 - 1} \cosh X)^n dX \text{ with } X_0 = \frac{1}{2} \ln \frac{z+1}{z-1}$$

Through the substitution $h = z - \sqrt{z^2 - 1} \cosh X$

$$Q_n(z) = \int_0^{1/\lambda} \frac{h^n}{\sqrt{1 - 2hz + h^2}} dh = \int_0^{1/\lambda} h^n (1 - h\lambda)^{-\frac{1}{2}} \left(1 - \frac{h}{\lambda}\right)^{\frac{1}{2}} dh$$

with

$$\lambda = z + \sqrt{z^2 - 1} \quad \frac{1}{\lambda} = z - \sqrt{z^2 - 1}$$

Putting $h = \frac{1}{\lambda}(1 - \omega)$ the integral representation of a hypergeometric series reads

$$\begin{aligned} Q_n(z) &= \int_0^1 \omega^{-\frac{1}{2}} (1 - \omega)^n \left(1 - \frac{\omega}{1 - \lambda^2}\right)^{\frac{1}{2}} \frac{1}{\lambda^n (\lambda^2 - 1)} \\ &= 2 \frac{2 \times 4 \dots 2n}{3 \times 5 \dots 2n + 1} \cdot \frac{1}{\lambda^n \sqrt{\lambda^2 - 1}} F\left(\frac{1}{2}, \frac{1}{2}; n + \frac{3}{2} \mid \frac{1}{1 - \lambda^2}\right) \end{aligned} \quad (70)$$

where F indicates a hypergeometric series with the variable $\lambda = \tilde{z}$

$$\begin{aligned} F\left(\alpha, \beta, \gamma, \frac{1}{\sqrt{1 - \tilde{z}^2}}\right) &= \sum_{\mu} \alpha_{\mu} \frac{1}{(1 - \tilde{z}^2)^{\mu}} = 1 + \frac{\alpha\beta}{\gamma \times 1!} \frac{1}{1 - \tilde{z}^2} \\ &+ \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)2!} \frac{1}{1 - \tilde{z}^2} + \dots \\ &+ \frac{\alpha(\alpha + 1)(\alpha + n)\beta(\beta + 1)(\beta + n)}{\gamma(\gamma + 1) \dots (\gamma + n)(n + 1)!} \frac{1}{(1 - \tilde{z}^2)^{n+1}} + \dots \end{aligned} \quad (71)$$

where $Q_n(\eta_0)$ is replaced in the series by $\tilde{z} = \eta_0 + \sqrt{\eta_0^2 - 1}$. It converges for $\tilde{z}^2 - 1 > 0$, $\eta_0^2 > 9/8$. This series converges very quickly for greater n and not too small η_0 . The $Q_n(\eta_0)$ with $n > 10$ are readily computable by this formula, for $n \leq 10$ the recursion is preferable.

The coefficients of the series

$$\alpha_\mu = \frac{\{1 \times 3 \dots [2(\mu - 1) + 1]\}^2}{\left(n + \frac{3}{2}\right) \left(n + \frac{5}{2}\right) \dots \left(n + \frac{2\mu + 1}{2}\right) 2^{2\mu} \mu!} \quad (72)$$

are computed by recursion for $n = 1, 2 \dots 19$ and $\mu = 1 \dots 7$. There is

$$\alpha_{\mu+1} = \alpha_\mu b_\mu$$

with

$$b_\mu = \frac{(2\mu + 1)^2}{(2\mu + 2)(2\mu + 2 + 2n + 1)}$$

The obtained numerical values are given in Table 2* along with the factors of the series for different n

$$A_n = 2 \frac{2 \times 4 \dots 2n}{3 \times 5 \dots 2n + 1}, A_{n+1} = \frac{2(n+1)}{2n+3} A_n \quad (73)$$

for $n = 1 \dots 19$ (table 3)

TABLE 3

FACTORS FOR THE SPHERICAL FUNCTIONS $Q_n(z)$

n	An	n	An
1	1.333 333 333	11	0.517 019 481
2	1.066 666 667	12	.496 338 702
3	.914 285 714	13	.477 955 787
4	.812 698 412	14	.461 474 553
5	.738 816 738	15	.446 588 277
6	.681 984 681	16	.433 055 299
7	.636 519 036	17	.420 682 290
8	.599 076 740	18	.409 312 499
9	.567 546 386	19	.398 817 307
10	.540 520 367		

*Table 2 may be found at the end of the text.

(d) Asymptotic summation of the series and approximation curves. - To be able to indicate approximation curves for $\frac{\partial}{\partial \eta} \sum (1)$ and $\frac{\partial}{\partial \eta} \sum (2)$ that pass through the computed points as closely as possible the limiting values for $\eta_0 \rightarrow 1$ and $\eta_0 \rightarrow \infty$ must be established first.

To this purpose consider the series $\sum (2)$ at $\eta = \eta_0$ (see equation 42)

$$\begin{aligned} \sum_{\eta=\eta_0}^{(2)} &= \sum (4n + 3) Q_{2n+1}(\eta_0) (1 - \xi^2) P'_{2n+1}(\xi) \\ &= 2\eta_0^2 \frac{1 - \xi^2}{(\eta_0^2 - \xi^2)^2} - \frac{1 - \xi^2}{\eta_0^2 - \xi^2} = \frac{1 - \xi^2}{(\eta_0^2 - \xi^2)^2} (\eta_0^2 + \xi^2) \end{aligned} \quad (74)$$

The derivative of the first series $\frac{\partial}{\partial \eta} \sum (1)$ is written

$$\frac{\partial}{\partial \eta} \sum (1) \bigg|_{\eta=\eta_0} = \sum (4n + 3) Q_{2n+1}(\eta_0) \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} \frac{Q_{2n+1}(\eta_0)}{\frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} (\eta_0^2 - 1) Q'_{2n+1}(\eta_0)} (1 - \xi)^2 P'_{2n+1}(\xi) \quad (75)$$

so that

$$\lim_{\eta_0 \rightarrow 1} \frac{Q_n(\eta_0)}{\frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} (\eta_0^2 - 1) Q'_n(\eta_0)} = -1$$

since

$$Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - f_{n-1}(z)$$

with

$$f_{n-1}(z) = \frac{2n+1}{1 \cdot n} P_{n-1}(z) + \frac{2n-5}{3(n-1)} P_{n-3}(z) + \frac{2n-9}{5(n-2)} P_{n-5}(z) + \dots$$

hence

$$(z^2 - 1)Q_n'(z) = -P_n(z) + (z^2 - 1) \left[\frac{1}{2} P_n'(z) \ln \frac{z+1}{z-1} - f'_{n-1} \right]$$

and

$$\frac{Q_n(z)}{\frac{1}{2} \ln \frac{z+1}{z-1} (z^2 - 1)Q_n'(z)} = \frac{1 - \frac{2f_{n-1}}{P_n(z) \ln \frac{z+1}{z-1}}}{1 - (z^2 - 1) \left[\frac{1}{2} \frac{P_n'(z)}{P_n(z)} \ln \frac{z+1}{z-1} - f'_{n-1} \right]} \rightarrow -1$$

f_{n-1} is a polynomial of the $(n-1)^{\text{th}}$ degree in z , hence finite

$$P_n(1) = 1; \ln \frac{z+1}{z-1} \xrightarrow{z=1} \infty; (z^2 - 1) \ln \frac{z+1}{z-1} \rightarrow 0$$

Thus the series (75) can be reduced to the series (74) multiplied by $\frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1}$:

$$\begin{aligned}
 \lim_{\eta_0 \rightarrow 1} \left[\frac{\partial}{\partial \eta} \sum (1) \right]_{\eta=\eta_0} &= - \lim_{\eta_0 \rightarrow 1} \frac{1 - \xi}{(\eta_0^2 - \xi^2)^2} (\eta_0^2 + \xi^2) Q_0 \xrightarrow[\xi=0]{\eta_0 \rightarrow 1} \left(- \frac{Q_0}{\eta_0^2} \right) \\
 &= - \lim_{\eta_0 \rightarrow 1} Q_0 \quad \text{that is } \frac{1}{Q_0} \frac{\partial}{\partial \eta} \sum (1) \Big/_{\substack{\eta=\eta_0 \\ \xi=0}} \xrightarrow{\eta_0 \rightarrow 1} -1
 \end{aligned} \tag{76}$$

The corresponding asymptotic expression for the other series is obtained by differentiation of (74) with respect to η_0 :

$$\begin{aligned}
 \frac{\partial}{\partial \eta_0} \left[\sum_{\eta=\eta_0} (2) \right] &= \sum (4n + 3) Q'_{2n+1}(\eta_0) (1 - \xi^2) P'_{2n+1}(\xi) \\
 &= (1 - \xi^2) \frac{2\eta_0(\eta_0^2 - \xi^2) - (\eta_0^2 + \xi^2) 4\eta_0}{(\eta_0^2 - \xi^2)^3} = (1 - \xi^2) \frac{-2\eta_0(\eta_0^2 + 3\xi^2)}{(\eta_0^2 - \xi^2)^3}
 \end{aligned}$$

This derivative is also uniformly convergent, since $\sum_{\eta=\eta_0}^{(2)}$ converges like the derivative of a power series. Multiplication by $(\eta_0^2 - 1)$ followed by another differentiation gives again a uniformly convergent series

$$\begin{aligned}
\frac{\partial^2}{\partial \eta_0^2} \left[\sum_{\eta=\eta_0}^{(2)} \right] &= \sum (4n+3)(2n+1)(2n+2) q_{2n+1}(\eta_0) (1-\xi^2) P'_{2n+1} \\
&= \frac{2(1-\xi^2)}{(\eta_0^2 - \xi^2)^4} \left[(\eta_0^4 + 3\xi^2)(\eta_0^2 - \xi^2) - 3\eta^2(\eta^2 + 3\xi^2)(1-\xi^2) \right. \\
&\quad \left. + 12\xi^2 \eta_0^2(\eta_0^2 - 1) \right] \xrightarrow{\xi=0} 2 \frac{\eta_0^2 - 3}{\eta_0^4}
\end{aligned} \tag{77}$$

As $q_{2n+1}(\eta_0) < q^{2n+1} \frac{1}{2n+2} \frac{1}{\eta_0 - 1}$, the convergence is again readily apparent. For the term of the series we get

$$a_n^* < \frac{4}{\eta_0 - 1} (2n+4)(2n+3)(2n+2) q^{2n+1}$$

The desired series reads then

$$\begin{aligned}
\frac{\partial}{\partial \eta} \sum_{\eta=\eta_0}^{(2)} &= \sum (4n+3)(2n+1)(2n+2) q_{2n+1}(\eta_0) \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} \\
&\quad \frac{q_{2n+1}(\eta_0)}{\frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} (\eta_0^2 - 1) q'_{2n+1}(\eta_0)} (1-\xi^2) P'_{2n+1}(\xi)
\end{aligned}$$

and owing to (77)

$$\lim_{\eta_0 \rightarrow 1} \left[\frac{\partial \sum^{(2)}}{\partial \eta} \right]_{\eta=\eta_0} = - \lim_{\eta_0 \rightarrow 1} \frac{1}{2} \ln \frac{\eta_0 + 1}{\eta_0 - 1} 2(1 - \xi^2)$$

$$\frac{\eta_0^4(\eta_0^2 - 3) + 3\xi^4(3\eta_0^2 - 1) + 2\xi^2 \eta_0^2(7\eta_0^2 - 9)}{\eta_0^2 - \xi^2} \xrightarrow{\xi=0} - \lim_{\eta_0 \rightarrow 1} Q_0 2 \frac{\eta_0^2 - 3}{\eta_0^4} = 4 \lim_{\eta_0 \rightarrow 1} Q_0 \quad (78)$$

hence

$$\frac{\frac{1}{Q_0} \frac{\partial \sum^{(2)}}{\partial \eta}}{\eta=\eta_0, \xi=0} \xrightarrow{\eta_0 \rightarrow 1} 4$$

The practical interpolation curves are (fig. 7)

$$\frac{\frac{\partial \sum^{(1)}}{\partial \eta}}{Q_0} \bigg/_{\eta=\eta_0} \approx - \frac{1}{2\eta_0^2} \left[1 + e^{3(1-\eta_0)} \right] \frac{\frac{\partial \sum^{(2)}}{\partial \eta}}{Q_0} \bigg/_{\eta=\eta_0} \approx \frac{3}{\eta_0^4} \left[1 + e^{20.64(1-\eta_0)} \right] - \frac{1}{\eta_0^2} \left[1 + e^{0.8177(1-\eta_0)} \right] \quad (79)$$

These curves actually have the given values for $\eta_0 = 1$ but the limiting values also coincide for $\eta_0 \rightarrow \infty$. Because for $\eta_0 \rightarrow \infty$ due to (70) $\tilde{z} = \lambda \rightarrow 2\eta_0$ and $F \rightarrow 1$

$$\eta_0 = \infty \qquad \eta_0 = \infty$$

$$Q_{2n+1}(\eta_0) \xrightarrow{\eta_0 = \infty} A_{2n+1} \frac{1}{\lambda^{2n+2}} \qquad (\eta_0^2 - 1)Q'_{2n+1}$$

$$= (2n+1)(\eta_0 Q_{2n+1} - Q_{2n}) \xrightarrow{\eta_0 = \infty} (2n+1) \left(A_{2n+1} \frac{\eta_0}{\lambda^{2n+2}} - A_{2n} \frac{1}{\lambda^{2n+1}} \right)$$

$$= (2n+1)A_{2n+1} \frac{1}{\lambda^{2n+2}} \left(\eta_0 - \frac{4n+3}{4n+2} 2\eta_0 \right)$$

$$= (2n+1)Q_{2n+1} \eta_0 - \frac{(2n+2)}{2n+1}$$

$$\text{thus } \frac{Q_{2n+1}^2}{(\eta_0^2 - 1)Q'_{2n+1}} \xrightarrow{\eta_0 = \infty} - \frac{1}{\eta_0(2n+2)} A_{2n} + 1 \frac{1}{\lambda^{2n+2}} - f(n) \frac{1}{\eta_0^{2n+3}}$$

If $\eta_0^2 \gg 1$ the second term in the series is already small compared to the first and it suffices to consider the first ($n = 0$):

$$(4n+3) \frac{Q_{2n+1}^2}{(\eta_0^2 - 1)Q'_{2n+1}} P'_{2n+1}(0) \xrightarrow[\eta_0 = \infty]{n=0} -3 \frac{1}{2\eta_0^2} - \frac{1}{(2\eta_0)^2} 1 = -\frac{1}{2\eta_0^2}$$

Since moreover, $Q_0 \xrightarrow{\eta_0 = \infty} \frac{1}{\eta_0}$ one obtains for $\xi = 0$

$$\frac{\frac{\partial}{\partial \eta} \sum^{(1)}}{Q_0} \bigg/_{\eta = \eta_0, \eta_0 = \infty} \longrightarrow -\frac{1}{2\eta_0^2} \qquad (80)$$

The terms of the other series merely differ by the factor $(2n + 1)(2n + 2)$, that is for $n = 0$

$$\frac{\frac{\partial}{\partial \eta} \sum^{(2)}}{Q_0} \bigg/_{\eta=\eta_0}^{\eta_0=\infty} = \frac{1}{\eta_0^2} \quad (81)$$

Translated by J. Vanier
National Advisory Committee
for Aeronautics

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TABLE 2

COEFFICIENTS FOR THE HYPERGEOMETRIC SERIES TO COMPUTE THE SPHERICAL FUNCTIONS

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7
1	0.100000000	0.032142857	0.014880952	0.008285985	0.005162806	0.003470553	0.002464384
2	.071428571	.017857143	.006764069	.003186917	.001720935	.001020751	.000648522
3	.055555556	.011363636	.003642191	.001487228	.000708620	.000376066	.000216174
4	.045454545	.007867133	.002185315	.000787356	.000335662	.000161171	.000084590
5	.038461538	.005769231	.001414027	.000455838	.000175823	.000077082	.000037220
6	.033333333	.004411765	.000967492	.000282185	.000099378	.000040082	.000017920
7	.029411765	.003482972	.000691066	.000184034	.000059627	.000022268	.000009269
8	.026315789	.002819549	.000510788	.000125143	.000037543	.000013054	.000005083
9	.023809524	.002329193	.000388199	.000088064	.000024597	.000008001	.000002927
10	.021739130	.001956522	.000301932	.000063770	.000016662	.000005091	.000001756
11	.020000000	.001666667	.000239464	.000047313	.000011613	.000003346	.000001092
12	.018518519	.001436782	.000193116	.000035843	.000008295	.000002261	
13	.017241379	.001251390	.000158004	.000027651	.000006053	.000001565	
14	.016129032	.001099707	.000130918	.000021672	.000004501	.000001107	
15	.015151515	.000974026	.000109688	.000017227	.000003403	.000000798	
16	.014285714	.000868726	.000092813	.000013865	.000002612	.000000585	
17	.013513514	.000779626	.000079230	.000011286	.000002031	.000000436	
18	.012820513	.000703565	.000068175	.000009279	.000001599	.000000329	
19	.012195122	.000638117	.000059085	.000007700	.000001273		

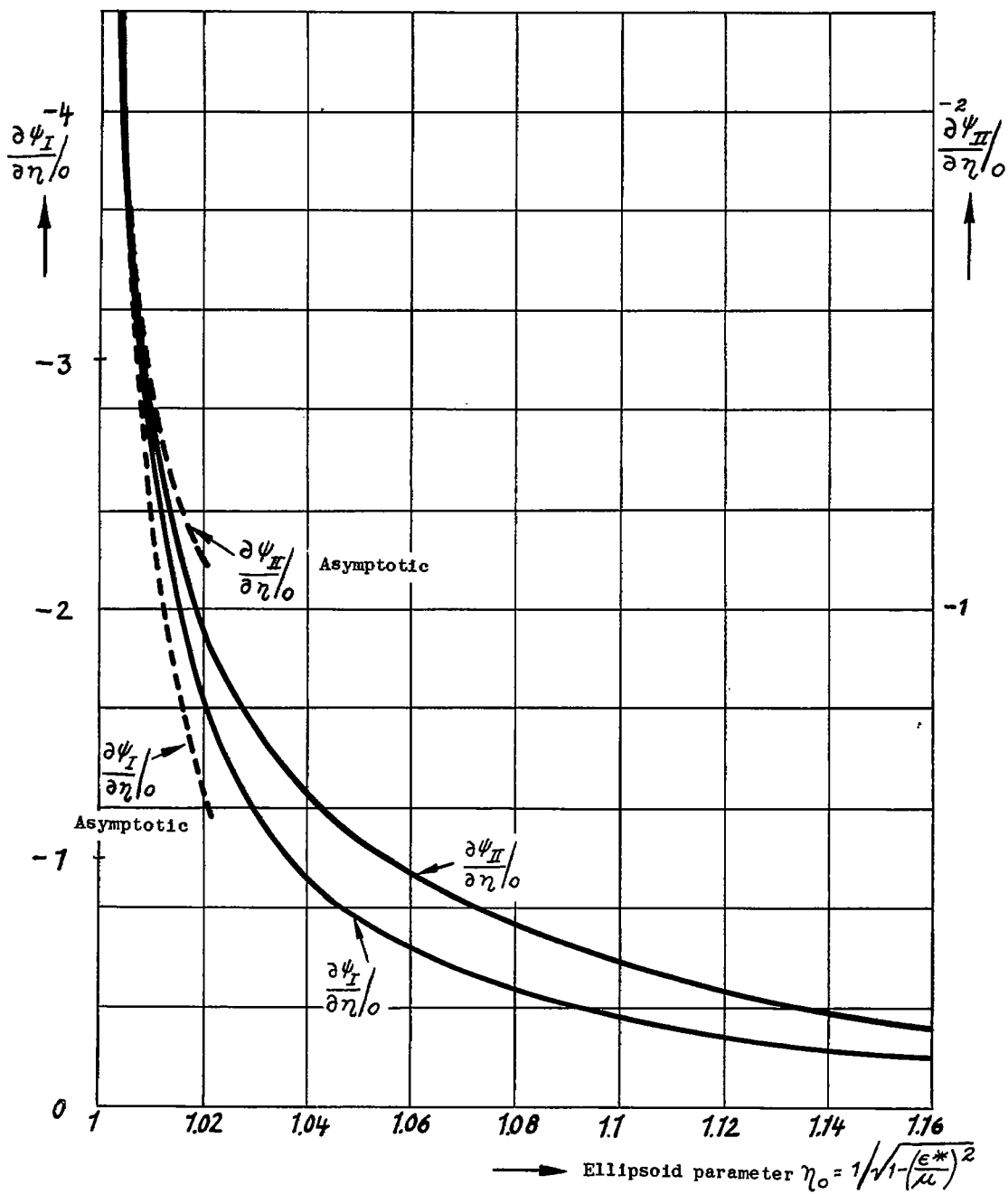


Figure 1.- $\frac{\partial \psi_I}{\partial \eta} /_0 = \left[\frac{\partial \psi_I}{\partial \eta} \right]_{\eta=\eta_0}$ and $\frac{\partial \psi_{II}}{\partial \eta} /_0 = \left[\frac{\partial \psi_{II}}{\partial \eta} \right]_{\eta=\eta_0}$

as a function of η_0 .

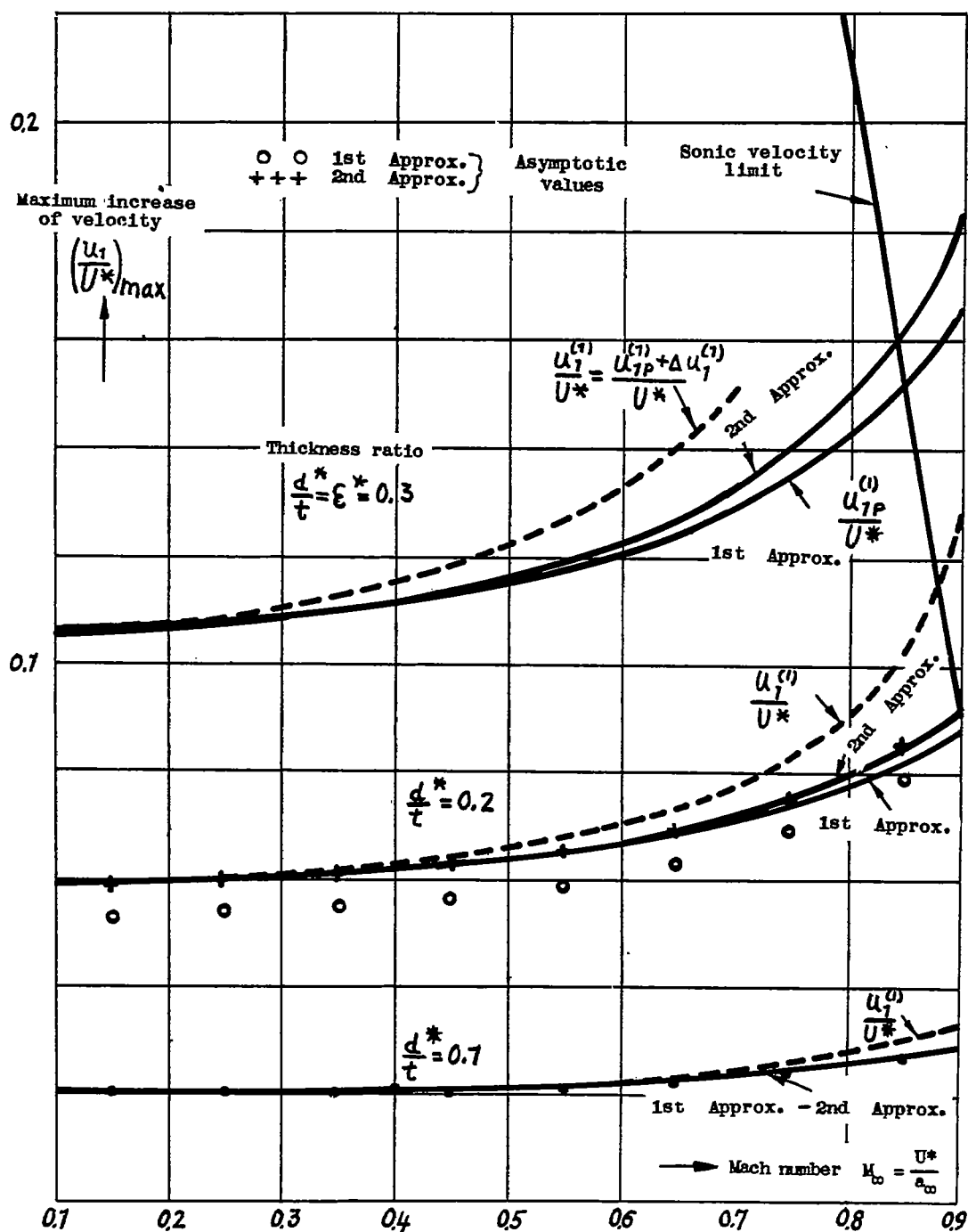


Figure 2.- Maximum excessive speed $(u_1/U^*)_{\max}$ on ellipsoids of the thickness ratio $\frac{d^*}{t} = \epsilon^*$ for various Mach numbers $M = \frac{U^*}{a_\infty}$.

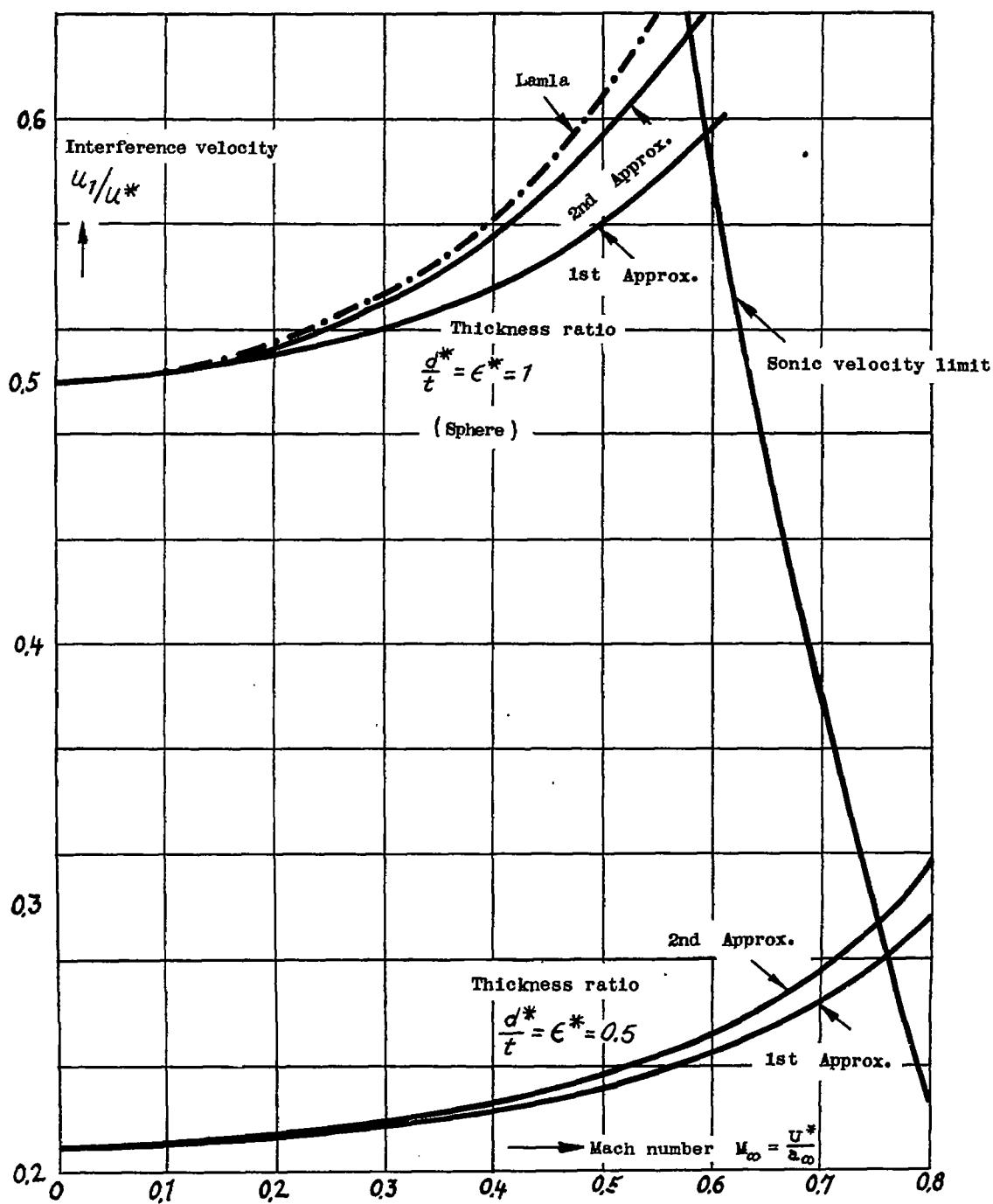


Figure 3.- Disturbance velocities on ellipsoid of first and second approximation. Comparison with Lamla's values for the sphere $d^*/t = \epsilon^* = 1$.

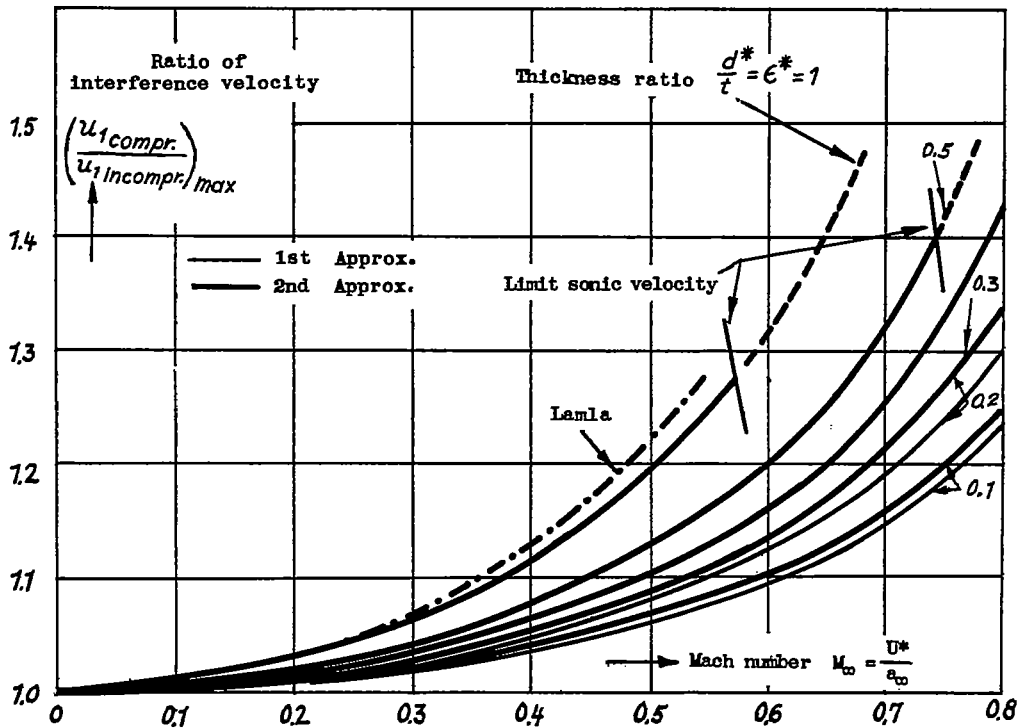


Figure 4.- Ratio of the excessive velocity $u_1 = u - U^*$ compressible to incompressible, first and second approximation.

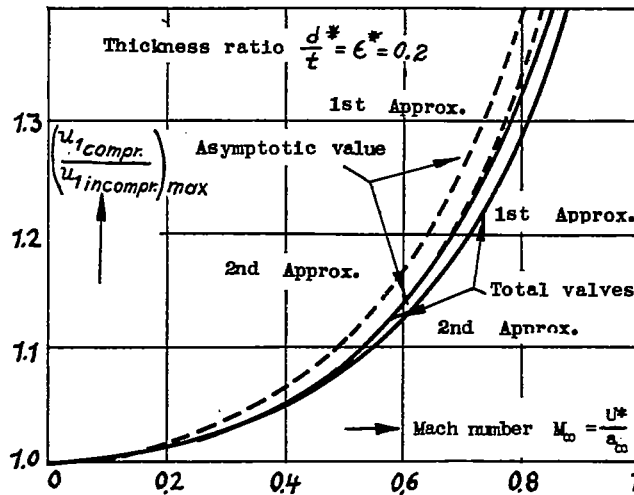


Figure 5.- Cutout from figure 4: $d^*/t = \epsilon^* = 0.2$. Comparison with the asymptotic values.

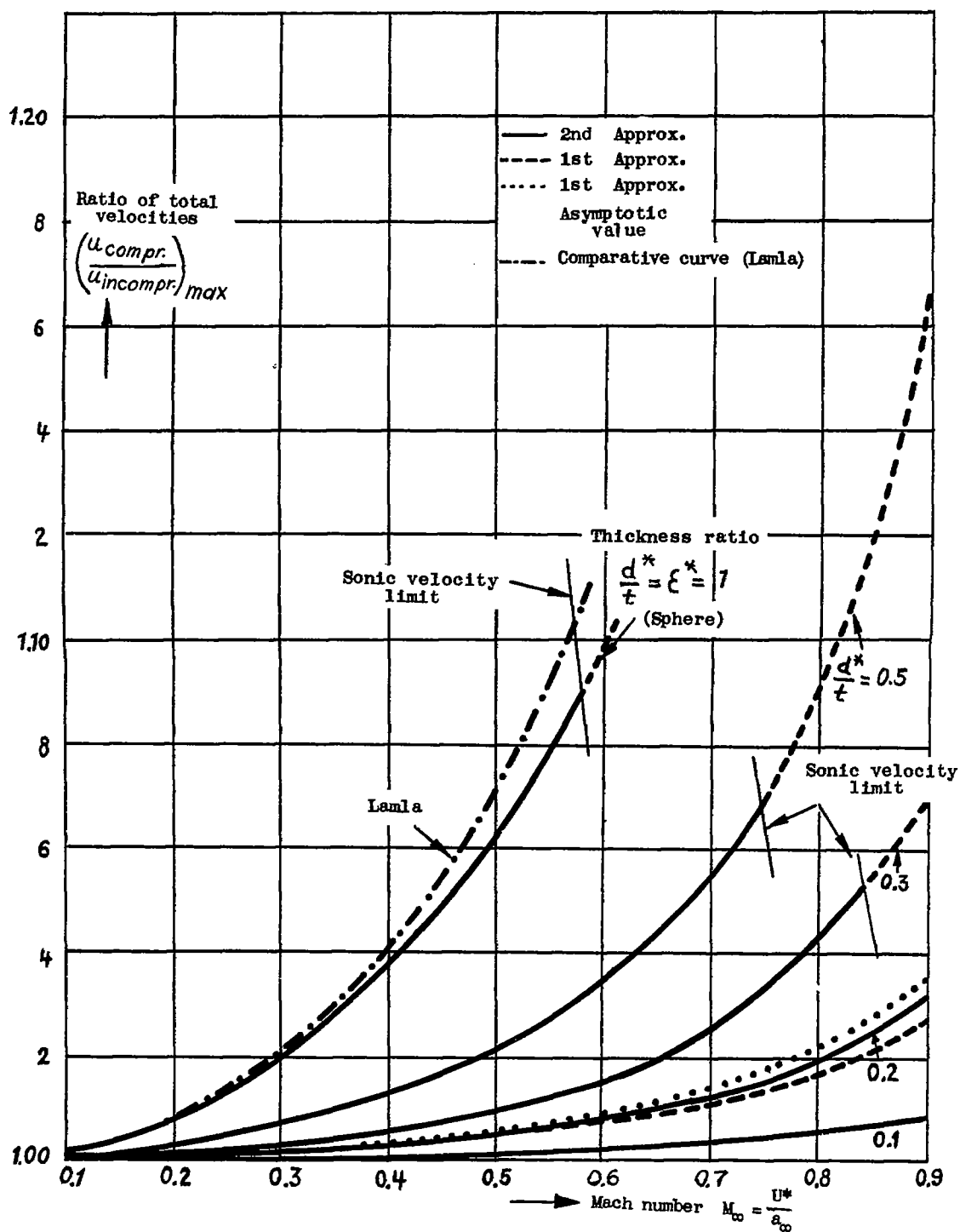


Figure 6.- Ratio of the total velocities $u = U^* + u_1$ compressible to incompressible as a function of the Mach number M_{∞} (Second approximation).

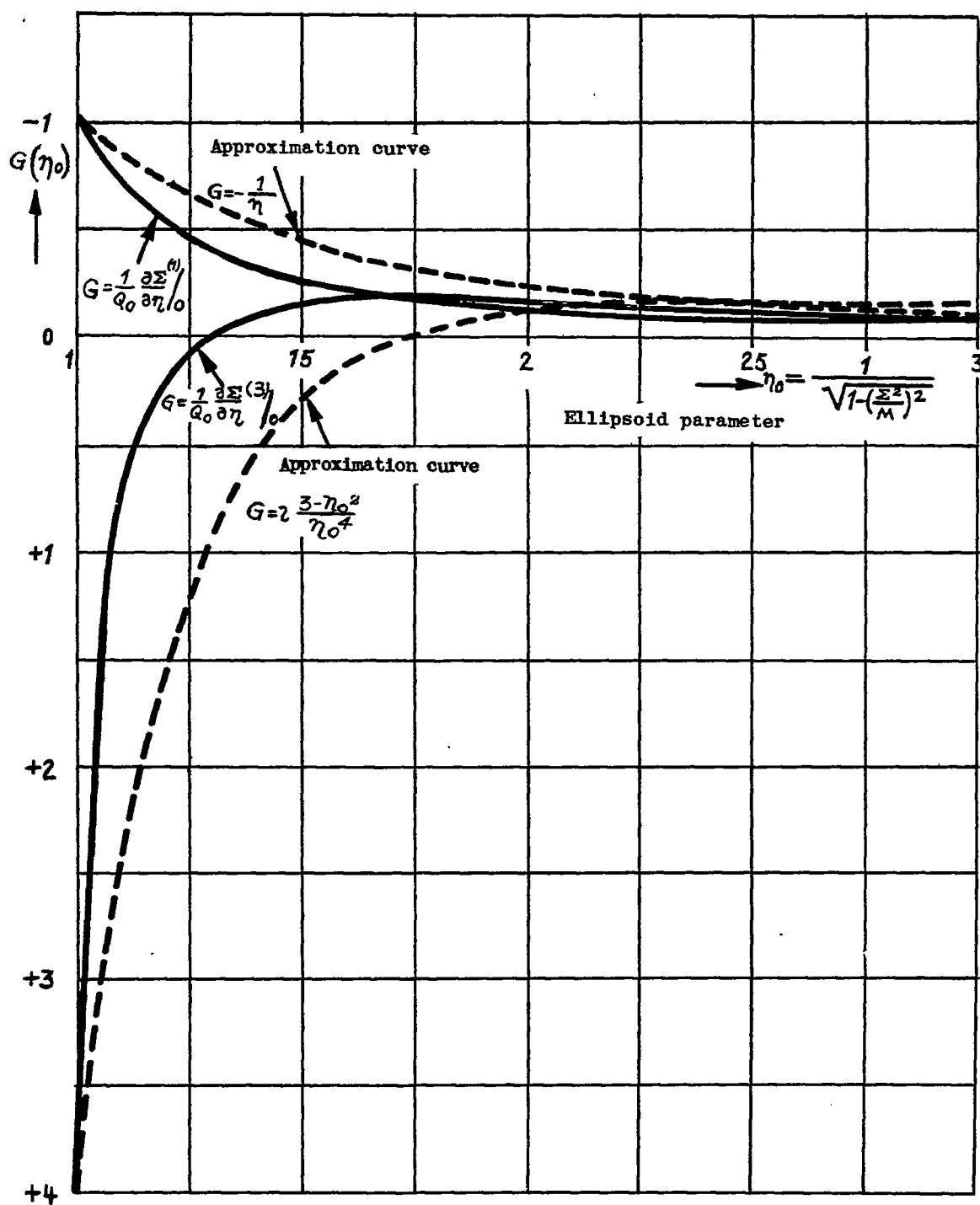


Figure 7.- Representation of the infinite series $\frac{\partial}{\partial \eta} \Sigma^{(1)}$ and $\frac{\partial}{\partial \eta} \Sigma^{(2)}$ by thin approximation curves (calculation values +) and comparison with the (dashed) approximation curves. $G(\eta_0) = \frac{1}{Q_0} \frac{\partial}{\partial \eta} \Sigma / \eta = \eta_0$.

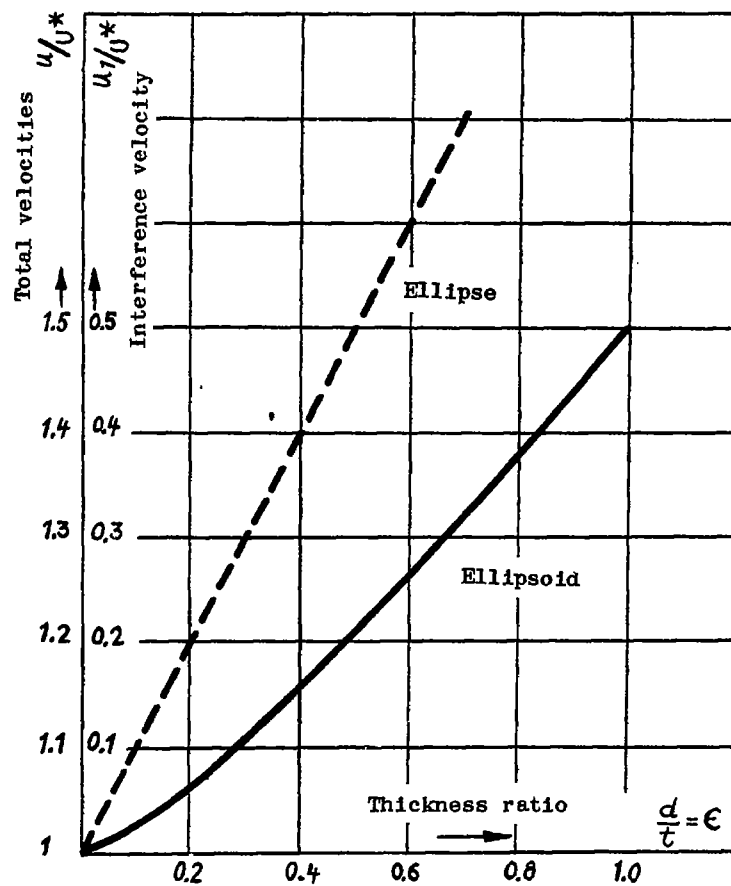


Figure 8.- Maximum velocity on the ellipsoid of the thickness ratio $\frac{d}{t}$ for incompressible flow ($M_\infty = 0$).